THE 'EXTENDED' ATKINSON FAMILY: THE CLASS OF MULTIPLICATIVELY DECOMPOSABLE INEQUALITY MEASURES, AND SOME NEW GRAPHICAL PROCEDURES FOR ANALYSTS

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Abstract. This paper introduces and characterises a class of inequality measures which extends the Atkinson family. This class contains canonical forms of all aggregative inequality measures, each bounded above by one, provides a new dominance criterion for ordering distributions in terms of inequality and offers some new graphical procedures for analysts. The crucial axiom for the characterisation is an alternative to the standard additive decomposition property that we call 'multiplicative decomposability', where the within-group component is a generalised weighted mean with weights summing exactly to one.

1. Basic intuition

Many of the more interesting applications of inequality measures in a population classified by characteristics such as age, gender, race or area of residence rely heavily on decomposable measures. However, any attempt to propose a theoretical framework to decompose inequality measures involves a number of conceptual experiments each of which results in different decompositions, and there may be no obvious general criterion for choosing between these alternatives. In most applications, however, the resulting inequality comparisons may at times conflict because 'replacing one index by another will … almost always change the relative significance of the between and within group terms' (Shorrocks [17]).

Intuition suggests that inequality in a population split into groups arises from the unequal distribution of income at two levels: the first is the unequal distribution of total income 'between' the groups, and the second is the unequal distribution of income 'within' each group.

Hence, in the decomposability of an inequality measure three distinct features must be tackled: i) the definition of the between-group component, ii) the definition of the within-group component, and iii) the way in which those components combine to yield in overall inequality.

When the between-group component is defined as the inequality level of a hypothetical distribution in which each person's income is replaced by the mean income of his/her subgroup, the within-group component is a weighted sum of the subgroup inequality levels and just the sum of the between- and within-group components is the method of aggregation, the only measures that meet the decomposability property are the Generalised Entropy indices (Shorrocks [17], Bourguignon [6], Cowell [7]).

Foster and Shneyerov [11] generalise the definitions of the between- and within-group components using a generalised mean in place of the arithmetic mean and characterise a two-parameter class of inequality measures containing, among other families of measures, the Generalised Entropy indices, the variance of logarithms and the path independent indices (Foster and Shneyerov [12]) which are to be introduced in the next paragraph.

An important problem arises when the within- and between- components cannot be formulated independently of each other. Shorrocks [17] and Anand [3] highlight this problem, stressing that changes in the between inequality can produce modifications not only in the between-group component but also in the within-group one, even though there may have been no change in within-group inequality. In this regard Foster and Shneyerov [12] explore an additive decomposition property that they call 'path independent decomposability', which requires that between- and within- group components are independent, and characterise the class of inequality measures that meet this property, called the path independent indices.

The Atkinson indices (Atkinson [4]) are not additively decomposable, and so obviously do not belong to the two-parameter class of Foster and Shneyerov [11]. However they have a very intuitive interpretation because of their highly functional forms. Any Atkinson index allows us to make explicit value judgements through the family parameter, and its numerical value has clear meaning. An index value of 0.4 means that if incomes were equally distributed, then 40% of total income is wasted, in the sense that we should need only 60% of the present total income to achieve the same level of social welfare. This suggests that these normatively significant indices serve better in practice than just descriptive ones. But as Atkinson indices are not additively decomposable, they have lost popularity in practical studies, in which sources of and variation in inequality need to be known and analysed.

We pause here to examine two features that specifically characterise additive decomposition. The first has to do with the within-group component definition as a weighted sum of the group inequality levels. One serious objection to the decomposition weights is that they usually do not sum to one. As a result, in a population split into subgroups, the within component is likely to be different even if the inequality levels in all the subgroups are exactly the same. Thus, in order to better interpret the within-group component as a measure of the unequal distribution of income within each group, in this paper the weights in the within-group component are required to sum to one.

Moreover, we propose a broader definition of the within component. Consider a simple example. Imagine two populations each split into two subgroups, each of the subgroups with half the population and income. Imagine that in the first population the subgroup inequality levels are 0.2 and 0.8 respectively. In the second they are 0.5 in both subgroups. As the weighted means of subgroup inequalities are the same in both populations, according to the traditional procedure the within-group inequality would also be the same. However, the situation is obviously not the same. In fact, in a real sense the inequality in the first situation is higher than in the second and, if this is so, it would be worth being reflected. The generalised means would allow us to shed light on such situations, which is why we suggest using the generalised means in the definition of the within-group component with weights summing to one.

The second issue in the decomposition formula we want to analyse has to do with the way in which the two components are aggregated. More often than not, multiplication is involved in the decomposition of many economic indices, such as decomposition of inequality indices by income sources, poverty indices and so on, and when this occurs, it allows us to evaluate the impact of marginal changes in a given component on the overall index. Indeed, the multiplicative decomposition can be transformed through the logarithmic transformation, so that it is additive in a simple form, permitting the overall percentage rate of change in inequality to be expressed as the sum of the percentage changes in the within- and the between- terms. Thus, the multiplicative decomposition can be very convenient in a dynamic context to account for changes in the overall inequality in terms of the changes inside the within- and between- components.

Having reached this point, nevertheless, it must be borne in mind that if the decomposition concerns an inequality index normalised at zero, decomposing overall inequality as the product of between- and within-inequality terms makes little sense. Indeed, under this assumption if the between- (within-) group component were zero so would be the overall inequality, even if there were inequality within (between) the subgroups, in direct conflict with the normalisation principle, which requires inequality to be zero only when the entire distribution is completely equal. In other words, multiplicative decomposition requires a definition in terms of equality components, which although initially a drawback, turns out to be the solution.

To sum up, the decomposition property proposed in this paper, which we call ‘multiplicative decomposition’, has the following features:

i) the between-group component is the equality level of a hypothetical distribution in which each person's income is replaced by the mean income of his/her group,

ii) the within-group component is a weighted generalised mean of the group equality levels, where the weights depend on their aggregated characteristics and their sum is equal to one, and

iii) overall equality is the product of within- and between-group equality terms.

Now it should be stressed that every index of the Atkinson family satisfies this multiplicative decomposition property (Lasso de la Vega and Urrutia [15]). Blackorby et al. [5] also present a multiplicative decomposition for this family in terms of equality indices using a welfare theory approach. The major difference between their proposal and ours is that they use subgroup 'equally distributed equivalent' income levels to determine the between group component of overall equality whereas we retain the traditional 'subgroup mean income' approach in the between group definition in our paper.

The core endeavour of this paper is to seek other inequality measures which share this multiplicative decomposition with the Atkinson family. Hence the rest of the paper focuses on offering a complete characterisation of the class of relative inequality measures that satisfy this form of decomposition. We begin by providing a natural extension of the Atkinson family which contains canonical forms of all aggregative measures, each bounded above by one, has a useful and intuitive geometric interpretation and can be used as a tool for ordering distributions in terms of inequality, in a way that is similar but not equivalent to Lorenz dominance. Then the paper shows that this extended Atkinson family is essentially the only class of continuous multiplicatively decomposable measures. Finally, the paper examines the implications of a path independence property of the type explored by Foster and Shneyerov [12] and shows that only one index in the extended family has a 'path independent' multiplicative decomposition.

The paper is structured as follows. The next section presents the notation and the definitions used in the paper. In Section 3 we introduce the one parameter extended family of Atkinson inequality measures, discuss its properties and provide some new graphical procedures for analysts. Our characterisation results are presented in Section 4. Finally, Section 5 offers some concluding remarks.

2. Notation and properties of inequality measures

We consider a population consisting of $n \geq 2$ individuals. Individual i's income is denoted by $y_i \in \mathbb{R}_{++} = (0, \infty)$, i=1,...,n. An income distribution is represented by a vector $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}^n_{++}$. We let $D = \bigcup_{\mathbf{n}=1}^{\infty} \mathbb{R}^n_{++}$ represent the set of all finite dimensional income distributions and denote the mean and population size of any $\mathbf{y} \in D$ by $\mu(\mathbf{y})$ and $n(\mathbf{y})$, respectively.

We say that distribution $\mathbf{x} \in D$ is a permutation of $\mathbf{y} \in D$ if $\mathbf{x} = \Pi\mathbf{y}$ for some permutation matrix Π; that $\mathbf{x}$ is an m-replication of $\mathbf{y}$ if $\mathbf{x} = (\mathbf{y}, \mathbf{y}, \ldots, \mathbf{y})$ and $n(\mathbf{x}) = m\, n(\mathbf{y})$ for some positive integer m; and that $\mathbf{x}$ is obtained from $\mathbf{y}$ by a progressive transfer if there exist i, j such that $x_i - y_i = y_j - x_j > 0; x_j > x_i;$ and $x_k = y_k$ for all $k \neq i, j$. We use the vector $\bar{\mathbf{y}}$ to signify the equalised version of $\mathbf{y}$, defined by $n(\bar{\mathbf{y}}) = n(\mathbf{y})$ and $\bar{y}_i = \mu(\mathbf{y})$ for all $i = 1, \ldots, n(\mathbf{y})$.

An inequality index I is a real valued continuous function $I : D \rightarrow \mathbb{R}$ and for the purpose of this paper we take the equality index as $E(\mathbf{y}) = 1 - I(\mathbf{y})$, which is a reasonable measure even if it takes negative values.

Suppose that the population of n individuals is split into $J \geq 2$ mutually exclusive subgroups with income distribution $\mathbf{y}^j = \left(y_1^j,...,y_{n_j}^j\right)$, mean incomes $\mu_j = \mu\left(\mathbf{y}^j\right)$ and population sizes $n_j = n\left(\mathbf{y}^j\right)$ for all j=1,...,J. Let inequality and equality in group j be written $I_j = I\left(\mathbf{y}^j\right)$ and $E_j = E\left(\mathbf{y}^j\right)$ respectively. Let p_j and s_j be the respective proportions of population and income of subgroup j. Following Foster and Shneyerov [12] the smoothed distribution associated with $\left(\mathbf{y}^1,...,\mathbf{y}^J\right)$ is defined by $\mathbf{y}^B = \left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right)$, in which each person's income is replaced by the mean income in his/her subgroup. The standardized distribution associated with $\left(\mathbf{y}^1,...,\mathbf{y}^J\right)$ is defined by $\mathbf{y}^W = \mu(\mathbf{y})\left(\mathbf{y}^1/\mu_1,...,\mathbf{y}^J/\mu_J\right)$ which rescales each group distribution so that the mean incomes of all subgroups equal the global mean income[1].

Certain properties which can be considered to be inherent to the concept of inequality have come to be accepted as basic properties for an inequality measure. They are listed below.

PROPERTY I. *Symmetry*. $I(\mathbf{x}) = I(\mathbf{y})$ whenever **x** is a permutation of **y**.

PROPERTY II. *Pigou-Dalton Transfers Principle.* $I(\mathbf{x}) < I(\mathbf{y})$ whenever **x** is obtained from **y** by a progressive transfer.

PROPERTY III. *Normalisation*. $I\left(\overline{\mathbf{y}}\right) = 0$ for all $\mathbf{y} \in D$.

PROPERTY IV. *Replication Invariance.* $I(\mathbf{x}) = I(\mathbf{y})$ whenever **x** is a replication of **y**.

An inequality index is consistent with a concept of relative inequality if it satisfies the following property:

PROPERTY V. *Scale Invariance.* $I(\lambda\mathbf{y}) = I(\mathbf{y})$ for all $\lambda > 0$.

Finally, Shorrocks (1984) introduced the following property for any partitioning of the population into disjoint subgroups:

[1]Actually, Foster and Shneyerov [12] introduce a generalisation of these definitions including representative incomes in place of mean incomes, but their definitions go beyond the scenario we consider in this paper.

PROPERTY VI. *Aggregative Principle.* (Shorrocks [18]) An inequality index I will be said to be aggregative if there exits an "aggregator" function Q such that $I(\mathbf{x},\mathbf{y}) = Q\big(I(\mathbf{x}), I(\mathbf{y}), \mu(\mathbf{x}), \mu(\mathbf{y}), n(\mathbf{x}), n(\mathbf{y})\big)$ for all $\mathbf{x},\mathbf{y} \in D$, where Q is continuous and strictly increasing in its first two arguments.

The class of all possible relative inequality measures satisfying the precedent properties is still rather large. Two prominent examples are widely used in the literature. The Generalised Entropy class, henceforth GE class, (Bourguignon [6], Shorrocks [17], [18], Cowell [7], [8]) is given by:

$$I_\alpha^{GE}(\mathbf{y}) = \begin{cases} \sum_{1\le i\le n}\left((y_i/\mu)^\alpha - 1\right) \big/ n\left(\alpha^2 - \alpha\right) & \alpha \ne 0,1 \\ -\sum_{1\le i\le n} \log(y_i/\mu)/n & \alpha = 0 \\ \sum_{1\le i\le n} (y_i/\mu)\log(y_i/\mu)/n & \alpha = 1 \end{cases} \qquad (1)$$

The other familiar example is provided by the Atkinson family (Atkinson [4]) which takes the form

$$I_\alpha^{A}(\mathbf{y}) = \begin{cases} 1 - \left(\left(\sum_{1\le i\le n}(y_i/\mu)^\alpha / n\right)^{1/\alpha}\right) & \alpha < 1 \quad \alpha \ne 0 \\ 1 - \left(\prod_{1\le i\le n}(y_i/\mu)^{1/n}\right) & \alpha = 0 \end{cases} \qquad (2)$$

It may be interesting to note that, as is well-known, these two families are monotonically related, in that, I_α^A can be obtained from I_α^{GE} via the following transformation:-

$$I_\alpha^{A}(\mathbf{y}) = F\left(I_\alpha^{GE}(\mathbf{y})\right) = \begin{cases} 1 - \left(1 + \left(\alpha^2 - \alpha\right) I_\alpha^{GE}(\mathbf{y})\right)^{1/\alpha} & \alpha < 1 \quad \alpha \ne 0 \\ 1 - e^{-I_0^{GE}(\mathbf{y})} & \alpha = 0 \end{cases}$$

In our approach to the within-group component definition, the class of generalised means (Hardy et al. [13]) will play an important role. Let $y_1,\ldots,y_n$ be real numbers and f an invertible function defined over the real numbers. The generalised mean is defined as $M_{f,\boldsymbol{\omega}}(\mathbf{y}) = f^{-1}\left(\sum_{1\le i\le n} \omega_i f(y_i)\right)$. where $\omega_j \ge 0$ for all j = 1,2,...n. Well-known examples of generalised means are the p-order means, denoted by $\mu_p(\mathbf{y})$, that is: $\mu_p(\mathbf{y}) = \left(\left(\sum_{1\le i\le n} y_i^p\right)/n\right)^{1/p}$ for $p \in \mathbb{R}, p \neq 0$ and $\mu_0(\mathbf{y}) = \left(\prod_{1\le i\le n} y_i\right)^{1/n}$, whence in particular $\mu_1(\mathbf{y})$ is the arithmetic mean and $\mu_0(\mathbf{y})$ is the geometric mean. The mapping $p \to \mu_p$ is a non-decreasing continuous function on all of $\mathbb{R}$. The limiting case at one extreme is as $p \to -\infty$, giving $\mu_p(\mathbf{y}) \to \min_i(y_i)$. At the other extreme, as $p \to \infty$, $\mu_p(\mathbf{y}) \to \max_i(y_i)$.[2]

As already mentioned, the aim of this paper is to explore a framework for inequality decomposition which broadens the standard definition of the within-group component and uses the product as a method of aggregation. In precise terms the multiplicative decomposition proposed in this study may be formulated as follows:

PROPERTY VII. *Multiplicative Decomposability.* An inequality index I is said to be *multiplicatively decomposable* if there exists a generalised mean such that for any exhaustive collection of $J \ge 2$ mutually exclusive subsets indices j=1,...,J, and for any distribution $\mathbf{y} \in D$, for the respective equality indices it is possible to write

$$E(\mathbf{y}) = M_{f,\boldsymbol{\omega}}(E_1,...,E_J)\, E\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) \tag{3}$$

where the weights ω_j depend only on the subgroup means and sizes with $\omega_j = \omega(p_j, s_j) \ge 0$ and $\sum_{1\le j\le J} \omega_j = 1$.

The first term on the right-hand side of this equation (3), which summarizes equality within the J population groups, will be denoted by $E_W(\mathbf{y})$. Notice that any multiplicatively decomposable measure obviously satisfies the aggregative principle (property VI).

This approach has several advantages. First, with respect to the within-group component definition, it clearly includes the traditional form of the within-group component provided the decomposition coefficients sum to one. Hence our formulation broadens the set of

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[2] See for example Steele [20].

possibilities to include the generalised means of the subgroup equality levels with weights summing to one. In this regard, the major advantage of this approach is that if the level of equality coincides in all groups, the within-group component is exactly that figure.

Second, as pointed out above, the multiplicative decomposition is a starting point for the derivation of the impact of marginal changes in a given component on overall equality. Indeed the multiplicative decomposition of these indices can be transformed, through the logarithmic transformation, so that it is additive in a simple form. The marginal effects derived from multiplicative decomposition appear in the following equation

$$\Delta E(\mathbf{y}) = \Delta E_W(\mathbf{y}) + \Delta E\left(\overline{\mathbf{y}^1}, \ldots, \overline{\mathbf{y}^J}\right)$$

where in general $\Delta x = \ln x_t - \ln x_{t-1} \approx \left(x_t - x_{t-1}\right) / x_{t-1}$ approximates the percentage change in x.

This equation shows that the overall percentage rate of change in equality can be expressed as the sum of the percentage changes in the within- and the between- components. This result is in accordance with intuition. It makes sense that the higher a component's contribution is, the lower the impact of marginal changes in the component on equality will be. By contrast, this analysis cannot be carried out with additive decomposition.

Nevertheless, as mentioned in the introduction, an important problem arises when analysts try to determine the shares of overall inequality which can be attributed either to differences between subgroups or to inequalities within those subgroups when the within- and between- components cannot be formulated independently of each other. In this regard Foster and Shneyerov [12] explore an additive decomposition property that they call 'path independent decomposability', which ensures that an identical decomposition is obtained whether one uses the smoothed distribution to define the between-group term and lets the within-group term be the residual, or one uses the standardized distribution to define the within-group term and lets the between-group term be the residual. Consequently, path independence is equivalent to requiring that between- and within- group components are independent. Seeking to discover the implications of this in the context of multiplicative decomposition, we introduce the following property:

PROPERTY VII. *Path Independent Multiplicative Decomposability.* An inequality index I has a *path independent multiplicative decomposition* if for any exhaustive collection of $J \geq 2$ mutually exclusive subsets indices j=1,...,J, and for any distribution $\mathbf{y} \in D$, for the respective equality indices it is possible to write

$$E(\mathbf{y}) = E(\mathbf{y}^W)\, E\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) \tag{4}$$

A multiplicatively decomposable index has a path independent decomposition when the within-group component that meets the multiplicative decomposition coincides exactly with the equality measure of the standardized distribution.[3]

3. The 'Extended' Atkinson Family

We may now investigate the nature of the family of inequality measures that satisfy all the properties described above. Consider the following single parameter class of measures

$$I_\alpha^{A*}(\mathbf{y}) = \begin{cases} 1-\left(\mu_\alpha(\mathbf{y})/\mu(\mathbf{y})\right) & \alpha < 1 \\ 1-\left(\mu(\mathbf{y})/\mu_\alpha(\mathbf{y})\right) & \alpha > 1 \\ 1-\left(\prod_{1\leq i\leq n}(\mu/y_i)^{y_i/n\mu}\right) & \alpha = 1 \end{cases} \tag{5}$$

where, as before, $\mu_p(\mathbf{y}) = \left(\left(\sum_{1\leq i\leq n} y_i^p\right)/n\right)^{1/p}$ for $p \neq 0$ and $\mu_0(\mathbf{y}) = \left(\prod_{1\leq i\leq n} y_i\right)^{1/n}$.

The expressions for the corresponding equality indices $E_\alpha^{A*}(\mathbf{y})$ are simply the bracketed terms on the right hand side of (5), which terms are always positive.

[3] Lasso de la Vega and Urrutia [16], following Foster and Shneyerov [12], introduce a generalization of this property including representative incomes, and characterise the class of path independent multiplicatively decomposable measures.

3.1 BASIC PROPERTIES OF THIS FAMILY

Firstly, it is noteworthy that the Atkinson family arises from this class when $\alpha<1$. The following theorem shows that the rest of the measures of the family also fulfil good properties.

Proposition 1: For each $\alpha \in \mathbb{R}$, I_{α}^{A*} *is an inequality measure which satisfies Symmetry, the Pigou-Dalton Transfers Principle, Normalization, Replication Invariance, the Scale Invariance Principle, the Aggregative Principle and is bounded above by one.*

Proof. Since the Atkinson indices satisfy all these properties it suffices to prove that they hold for the members of this family for $\alpha \geq 1$. It is clear that all of these measures are bounded above by 1. Furthermore, we can mechanically transform one index $I_{\alpha}^{A*}(\mathbf{y})$ from the family (5) into the other, $I_{\alpha}^{GE}(\mathbf{y})$, from the GE family, for all $\mathbf{y} \in D$, using the following formulae

$$I_{\alpha}^{A*}(\mathbf{y}) = F\left(I_{\alpha}^{GE}(\mathbf{y})\right) = \begin{cases} 1-\left(\left(1+\left(\alpha^{2}-\alpha\right)I_{\alpha}^{GE}(\mathbf{y})\right)^{1/\alpha}\right)^{-1} & \alpha>1 \\ 1-e^{-I_{1}^{GE}(\mathbf{y})} & \alpha=1 \end{cases}$$

where it can easy be checked that, at any given α, $F:\mathbb{R}_{+} \to \mathbb{R}$ is a continuous strictly increasing function with F(0)=0. Then by a result in Shorrocks [18], I_{α}^{A*} satisfies the properties required. *Q.E.D.*

As already mentioned, the Atkinson family is ordinally equivalent to one tail of the GE family. Now from the proof of this proposition it can be seen that each member of the GE family for all values of the parameter is an increasing transformation of an index of the family (5) and can be transformed into it with the same type of monotonic transformation. This result provides a natural extension of the Atkinson indices which may be referred to as *the extended*

Atkinson family. Moreover, this family contains alternative canonical forms of all continuous aggregative inequality measures, since there is a measure in this family corresponding to each continuous aggregative index which orders distributions in precisely the same way.

Figure 1. I_α^{A*} (y) as a function of the α-parameter.

Another interesting characteristic of this family is that the inequality value I_α^{A*} varies continuously as a function of the α-parameter for each income distribution, except when α=1. (As $\alpha \to 1$, I_α^{A*} (**y**) tends to the totally insensitive measure, whereas I_1(**y**) is ordinally equivalent to what is commonly called the Theil inequality index). Actually, for any given **y**, the family has two tails, according to whether α is less or greater than 1. An example is given in Figure 1. The α-parameter is clearly a measure of the degree of relative sensitivity to transfers at different income levels. As α increases I_α^{A*} (**y**) becomes more sensitive to transfers at the upper end than at the lower end and in the middle part of the distribution. The limiting case is as $\alpha \to \infty$, giving $I_\alpha^{A*}(\mathbf{y}) \to 1-\mu/\max_i(y_i)$, which only considers transfers to the richest income group. By contrast, as α decreases the opposite is true, in other words, this family becomes more sensitive to transfers at the lower end of the distribution. The limiting case is as $\alpha \to -\infty$, giving $I_\alpha^{A*}(\mathbf{y}) \to 1-\min_i(y_i)/\mu$, which only takes account of transfers to the very lowest income group. In fact, when α is less than 1, I_α^{A*} satisfies the transfer sensitivity principle according to Shorrocks and Foster [19].

3.2 I_α^{A*}-CURVE DOMINANCE

What value of α should we choose to determine the specific value of inequality? The answer is by no means simple because the resulting inequality comparisons may be sensitive to the choice of this value and may occasionally clash in significant aspects.

Apart from that, a standard procedure in order to avoid any conflict is to demand unanimous agreement among classes of inequality measures. To that end we now suggest the use of I_α^{A*}-curves as an appealing tool for ordering distributions: when the curve of one distribution lies everywhere above the curve of another, it displays unambiguously more inequality as measured by this family. However, since the members of this family, as mentioned above, can be considered as canonical forms of all continuous aggregative inequality indices, checking this I_α^{A*}-dominance enables us to establish inequality comparisons that necessarily hold for all continuous aggregative inequality indices.

The Lorenz curve is usually used to test whether one distribution is unambiguously more unequal than another providing that one accepts the principle of transfers, since Lorenz ordering is equivalent to inverse stochastic dominance to the second order. Therefore, Lorenz dominance obviously implies I_α^{A*}-curve dominance. However the ranking of distributions implied by Lorenz dominance and that of I_α^{A*}-curve dominance are not equivalent. To illustrate this point, take a six-person society and consider two distributions **x**:=(0.1, 0.4, 0.75, 0.75, 1, 3) and **y**:=(0.25, 0.25, 0.5, 1, 2, 2). According to the Lorenz criterion distributions **x** and **y** cannot be ranked since their Lorenz curves cross twice. By contrast it may be observed that distribution **y** dominates distribution **x** under I_α^{A*}-curve dominance since $I_\alpha^{A*}(\mathbf{x}) \geq I_\alpha^{A*}(\mathbf{y})$ for all $\alpha \in \mathbb{R}$, as shown in Figure 2.

Figure 2. I_α^{A*}-curves for the distributions x and y.

Bearing in mind that the aggregative principle has often been invoked for measuring inequality in a population split into groups, I_α^{A*}-curves provide a powerful tool for testing whether the unanimous agreement applied to the class of continuous aggregative indices leads to an unquestionable ruling when the Lorenz curves intersect.

Nevertheless, obviously, it is not rare in practice for I_{α}^{A*}-curves also to intersect. If one were interested in giving more weight to transfers at the lower end of the distribution than at the top, the measure should satisfy the transfer sensitivity principle, and it is possible under special circumstances to obtain a conclusive ranking when one refers to this class of measures. Whereas Shorrocks and Foster [19] demonstrated that third order stochastic dominance allows us to characterize unanimous agreement among measures of this class, Davies and Hoy [10] proved that a variance condition allows distributions whose Lorenz curves intersect to be ordered unambiguously among them. As regards I_{α}^{A*}-curve dominance, when the curves do not intersect for α less than or equal to 1, distributions may be ranked conclusively among the class of aggregative transfer sensitive inequality measures.

In any case, since the normative judgements associated with the values of α are both explicit and appealing this I_{α}^{A*}-curve dominance allows us to make inequality comparisons giving a fuller description of differences in inequality. If the curve of one distribution crosses that of another from above on the left-hand side of the graph, the first distribution is more unequal than the second according to large negative values of α, which are sensitive to the incomes of the people who are worst off. Conversely, when the curves intersect on the right-hand side with α large but positive the rule would be particularly sensitive to the incomes of the richest people. So, using I_{α}^{A*}-curves to make inequality comparisons rather than relying on summary indices alone not only becomes a useful tool for ranking distributions but also gives us an insight into how we might explain what is observed.

3.3 MULTIPLICATIVE DECOMPOSITION PROPERTY

The following proposition shows that the measures of the extended Atkinson family meet the multiplicative decomposition property.

Proposition 2: Let consider any exhaustive collection of $J \geq 2$ mutually exclusive population subgroups. For each $\alpha \in \mathbb{R}$, I_{α}^{A*} satisfies Multiplicative Decomposability in the following form:

$$E_{\alpha}^{A*}(\mathbf{y}) = \begin{cases} \left(\sum_{1\leq j\leq J} \omega_j \left(E_{\alpha,j}^{A*}\right)^{\alpha}\right)^{1/\alpha} E_{\alpha}^{A*}\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) & \alpha < 1 \quad \alpha \neq 0 \\ \left(\sum_{1\leq j\leq J} \omega_j \left(E_{\alpha,j}^{A*}\right)^{-\alpha}\right)^{-1/\alpha} E_{\alpha}^{A*}\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) & \alpha > 1 \\ \prod_{1\leq j\leq J} \left(E_{\alpha,j}^{A*}\right)^{\omega_j} E_{\alpha}^{A*}\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) & \alpha = 0,1 \end{cases} \tag{6}$$

where $\omega_j = p_j^{1-\alpha} s_j^{\alpha} \Big/ \sum_{j=1}^{J} p_j^{1-\alpha} s_j^{\alpha}$ for all α.

Proof: The proof is straightforward after a few lines of standard computations and rearrangements. *Q.E.D.*

Now it may be worth comparing the additive decomposition for the GE family with the multiplicative decomposition presented here. Note that this multiplicative decomposition plays a role symmetrical to the one played by additive decomposition. Indeed, for the GE indices the within-component is a weighted average of the inequality of each individual group and in the extended Atkinson family the within- term is a p-order weighted mean of the group equalities with $p \leq 1$. The decomposition coefficients for these indices are the same as in the additive decomposition but normalised and their sum is equal to 1. If the level of equality coincides in all groups, the mean and the p-order mean lead to the same result. The bigger the difference in the levels of equality of the groups, the smaller the p-order mean with p<1, so that the p-order mean indicates not only the mean levels of equality of groups but also the differences between those levels. In this sense, we consider that the p-order means have advantages over the arithmetic mean, especially in situations where the distributions within groups are very unequal.

Each multiplicatively decomposable measure also permits a simple geometric interpretation in a unit box, as shown in Figure 3.

Figure 3. Geometric interpretation of a multiplicatively decomposable measure for a given y

The comparison between the inequality levels of two income distributions is reduced to a comparison between their respective level curves and the corresponding projections on the axes, with information being provided at all times on how much progress has been made towards equality in each component, and how far there is still to go. For applied economists and policy analysts, this graphical approach can effectively convey information about inequality although great care is needed to interpret the empirical results. On the one hand, we must certainly reference here the interesting arguments of Kanbur [14]. He warns against normative use of decomposition findings because in his opinion they cannot determine the appropriate focus for policy interventions. Policy instruments that target between-group differences and those which target within-group differences must be costed and their benefits and effectiveness compared.

On the other hand, as already mentioned, a great deal of confusion appears when the within- and between-components are not path-independent. Shorrocks [17] proves that only one member of the GE family satisfies this requirement, and concludes that "I_0 is the most satisfactory of the decomposable measures, allowing total inequality to be unambiguously split into the contribution due to differences between subgroups and the contribution due to inequality within each subgroup". We find a similar result for the extended Atkinson family.

Proposition 3: I_0^{A*} *is the only measure of the extended Atkinson family which has a Path Independent Multiplicative Decomposition.*

Proof. First of all it is straightforward to prove that the equality measure of the standardized distribution can be calculated using the following formulae

$$E_\alpha^{A*}\left(\mathbf{y}^W\right)=\begin{cases}\left(\sum_{1\le j\le J} p_j\left(E_{\alpha,j}^{A*}\right)^{\alpha}\right)^{1/\alpha} & \alpha<1 \quad \alpha\neq 0\\ \left(\sum_{1\le j\le J} p_j\left(E_{\alpha,j}^{A*}\right)^{-\alpha}\right)^{-1/\alpha} & \alpha>1\\ \prod_{1\le j\le J}\left(E_{\alpha,j}^{A*}\right)^{p_j} & \alpha=0,1\end{cases}$$

Comparing these expressions with the within-group components given in (6) yields $p_j=\omega_j=p_j^{1-\alpha}s_j^{\alpha}\Big/\sum_{j=1}^{J}p_j^{1-\alpha}s_j^{\alpha}$ for all j , k = 1,…,J and for any exhaustive collection of $J\ge 2$ mutually exclusive subgroups,

which only holds when $\alpha=0$. Q.E.D.

4. The class of multiplicatively decomposable inequality measures

We now turn to the main aim of this paper which is to characterise the class of relative inequality measures that satisfy the multiplicative decomposition property. The following theorem shows that for each $\alpha\in\mathbb{R}$ there exists a unique inequality measure, up to a positive constant, satisfying the multiplicative decomposition. Particularly when we consider the constant equals 1 we find the extended Atkinson family (5). Thus the theorem shows that this family is completely characterised by the multiplicative decomposition, in other words, that the extended Atkinson family is essentially the only class of continuous multiplicatively decomposable measures.

Theorem: An inequality measure $I : D \to \mathbb{R}$ satisfies Symmetry, the Pigou-Dalton Transfers Principle, Normalization, Replication Invariance, the Scale Invariance Principle and Multiplicative Decomposition if and only if it takes the form

$$I(\mathbf{y}) = 1 - \left(\mu_\alpha(\mathbf{y})/\mu(\mathbf{y})\right)^K \quad \alpha < 1$$

or

$$I(\mathbf{y}) = 1 - \left(\mu(\mathbf{y})/\mu_\alpha(\mathbf{y})\right)^K \quad \alpha > 1 \qquad (7)$$

or

$$I(\mathbf{y}) = 1 - \left(\prod_{1 \le i \le n} (\mu/y_i)^{y_i/n\mu}\right)^K$$

where K is a positive constant.

Proof. Suppose that I satisfies Symmetry, the Pigou-Dalton Transfers Principle, Normalization, Replication Invariance, the Scale Invariance Principle and Multiplicative Decomposition. Then by Shorrocks [18] I is an increasing transformation of an index of the GE class (1). The proof of proposition 1 shows that each member of the GE family is, in turn, an increasing transformation of an index of the extended Atkinson family (5). Consequently there exists $\alpha \in \mathbb{R}$ and a continuous strictly increasing function $F: \mathbb{R}_+ \to \mathbb{R}$ such that

$$I(\mathbf{y}) = F\left(I_\alpha^{A*}(\mathbf{y})\right) \text{ for all all } \mathbf{y} \in D \qquad (8)$$

with F(0)=0. To complete this direction of the proof we now derive the form of F.

Rewriting (8) in terms of equality indices and denoting $G(x) = 1 - F(1-x)$ then we obtain

$$E(\mathbf{y}) = 1 - I(\mathbf{y}) = 1 - F\left(1 - E_\alpha^{A*}(\mathbf{y})\right) = G\left(E_\alpha^{A*}(\mathbf{y})\right) \qquad (9)$$

where $G: \mathbb{R}_{++} \to \mathbb{R}$ is also a continuous strictly increasing function.

Then, for any exhaustive collection of $J \ge 2$ mutually exclusive subgroups j=1,...,J, we get

$$E_j = G\left(E_{\alpha,j}^{A*}\right) \text{ for all j=1,...,J} \qquad (10)$$

and

$$E\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) = G\left(E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right)\right) \tag{11}$$

Multiplicative decomposability of I and I_α^{A*} assures that there exists generalised means for which we obtain

$$E(\mathbf{y}) = M_{f,\boldsymbol{\omega}}\left(E_1,...,E_J\right) E\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) \tag{12}$$

and

$$E_\alpha^{A*}(\mathbf{y}) = M_{f',\boldsymbol{\omega}'}\left(E_{\alpha,1}^{A*},...,E_{\alpha,J}^{A*}\right) E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) \tag{13}$$

with the weights summing to one.

Substituting (9), (10) and (11) into (12) yields

$$G\left(E_\alpha^{A*}(\mathbf{y})\right) = M_{f,\omega}\left(G\left(E_{\alpha,1}^{A*}\right),...,G\left(E_{\alpha,J}^{A*}\right)\right) G\left(E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right)\right) \tag{14}$$

Combining (13) and (14) we find

$$G\left(M_{f',\boldsymbol{\omega}'}\left(E_{\alpha,1}^{A*},...,E_{\alpha,J}^{A*}\right) E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right)\right) = M_{f,\omega}\left(G\left(E_{\alpha,1}^{A*}\right),...,G\left(E_{\alpha,J}^{A*}\right)\right) G\left(E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right)\right) \tag{15}$$

Consider any $x \in (0,1]$ and $y \in (0,1]$ and construct subgroup distributions satisfying $E_{\alpha,j}^{A*} = E_{\alpha,k}^{A*} = x$ for any j , k = 1,…,J and $E_\alpha^{A*}\left(\overline{\mathbf{y}^1},...,\overline{\mathbf{y}^J}\right) = y$. If we apply (15) to these particular distributions, the assumed properties of the generalised means ensure that

$$G(x\,y) = G(x)\,G(y) \qquad \text{for all } x, y \in (0,1] \tag{16}$$

Furthermore $G(x) = G\left(\sqrt{x}\sqrt{x}\right) = \left(G\left(\sqrt{x}\right)\right)^2 \geq 0$ for all $x, y \in (0,1]$

Following Aczél ([2], p. 17) there exists a unique function[4] $\widetilde{G}:\mathbb{R}_{++}\to\mathbb{R}$ which is an extension of G, in that $\widetilde{G}(x)=G(x)$ for all $x,y\in(0,1]$, and also satisfies

$$\widetilde{G}(x\,y)=\widetilde{G}(x)\widetilde{G}(y) \text{ for all } x,y\in\mathbb{R}_{++} \quad (17)$$

with $\widetilde{G}(x)>0$ for all $x,y\in\mathbb{R}_{++}$. Aczél ([1], p. 41) has shown that the general solution to this functional equation (17) satisfying the given properties is

$$G(x)=x^{K} \quad K>0 \quad (18)$$

which in turn from (8) and (9) yields

$$I(\mathbf{y})=F\left(I_{\alpha}^{A*}(\mathbf{y})\right)=1-\left(1-I_{\alpha}^{A*}(\mathbf{y})\right)^{K} \text{ with } K>0.$$

which completes one direction of the proof.

Conversely, suppose that I takes the form giving in (7) for some $\alpha\in\mathbb{R}$ and K>0. In other words, $I(\mathbf{y})=1-\left(1-I_{\alpha}^{A*}(\mathbf{y})\right)^{K}$ for some $\alpha\in\mathbb{R}$ and K>0. Obviously I can be transformed into I_{α}^{A*} using the following formula

$$I(\mathbf{y})=F\left(I_{\alpha}^{A*}(\mathbf{y})\right)=1-\left(1-I_{\alpha}^{A*}(\mathbf{y})\right)^{K} \text{ with K>0 for all } \mathbf{y}\in D.$$

where $F:\mathbb{R}_{++}\to\mathbb{R}$ is a continuous strictly increasing function with F(0)=0. Then once again by Shorrocks [18] I satisfies symmetry, the Pigou-Dalton transfers principle, normalisation, replication invariance, the scale invariance principle and the aggregative principle.

It remains only to verify multiplicative decomposition (property VI). Consider any exhaustive collection of $J\geq 2$ mutually exclusive subgroups j=1,...,J. After a few lines of standard computations and rearrangements it is straightforward to prove that the equality index E=1-I meets the multiplicative decomposition by using the following formulae

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[4] To do that let $x\in\mathbb{R}_{++}$ be arbitrary. Then there exists a positive integer n such that $x/n\in(0,1]$. We define $\widetilde{G}(x)=\widetilde{G}((x/n)n)=(G(x))^{1/n}$. This definition is unambiguous and $\widetilde{G}$ is an extension of G satisfying the functional equation (17).

$$E(\mathbf{y}) = \left(\sum_{1\le j\le J} \omega_j \left(E(\mathbf{y}^j)\right)^{\alpha/K}\right)^{K/\alpha} E\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) \quad \text{if } \alpha < 1 \text{ and } \alpha \neq 0$$

or

$$E(\mathbf{y}) = \left(\sum_{1\le j\le J} \omega_j \left(E(\mathbf{y}^j)\right)^{-\alpha/K}\right)^{-K/\alpha} E\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) \quad \text{if } \alpha > 1$$

or

$$E(\mathbf{y}) = \prod_{1\le j\le J} \left(E(\mathbf{y}^j)\right)^{\omega_j} E\left(\overline{\mathbf{y}^1},\ldots,\overline{\mathbf{y}^J}\right) \quad \text{if } \alpha = 0,1$$

with weights $\omega_j = p_j^{1-\alpha} s_j^{\alpha} \Big/ \sum_{j=1}^{J} p_j^{1-\alpha} s_j^{\alpha}$ for all α, which completes the proof. *Q.E.D.*

5. Concluding remarks

In this paper we have proposed an alternative multiplicative decomposition property, and have characterised a new class of inequality measures that we call the ‘extended’ Atkinson family containing the prominent Atkinson family. We have shown the relationship between this class and the Generalised Entropy indices. Indeed, it is well-known that the Atkinson family is ordinally equivalent to one tail of the Generalised Entropy family and the ‘extended’ Atkinson family answers the question of what happens with the other tail since now each member of the Generalised Entropy family can be transformed, with the same type of monotonic transformation, into one of this extended family. Consequently the ‘extended’ Atkinson family contains canonical forms of all aggregative measures, each bounded above by 1, allowing us to develop a dominance criterion that is similar but not equivalent to Lorenz dominance and some new graphical procedures to rank distributions using the class of aggregative indices. Some analysts may wish to adopt this family in applied works, at least, in order to check on the robustness of results implied by other inequality measures.

On the other hand each Atkinson index satisfies the transfer sensitivity principle, in other words, it gives more weight to transfers at the lower end of the distribution than at the top. The concern with inequality stems from the injustice of extremely low incomes, so the inequality measure must be sensitive to what happens to the poor. The other side of the same coin is the injustice of extremely high incomes. This suggests that sometimes it would be of interest to choose an inequality measure sensitive to such incomes in order to obtain a detailed explanation of what really happens. In this sense the extended Atkinson family, which contains specific measures sensitive to large incomes as well as the traditional low-income-sensitive Atkinson family, may be of use as a measuring tool that can be adjusted to be particularly sensitive to any income in the distribution.

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Figure 1. I_α^{A*} (y) as a function of the α-parameter

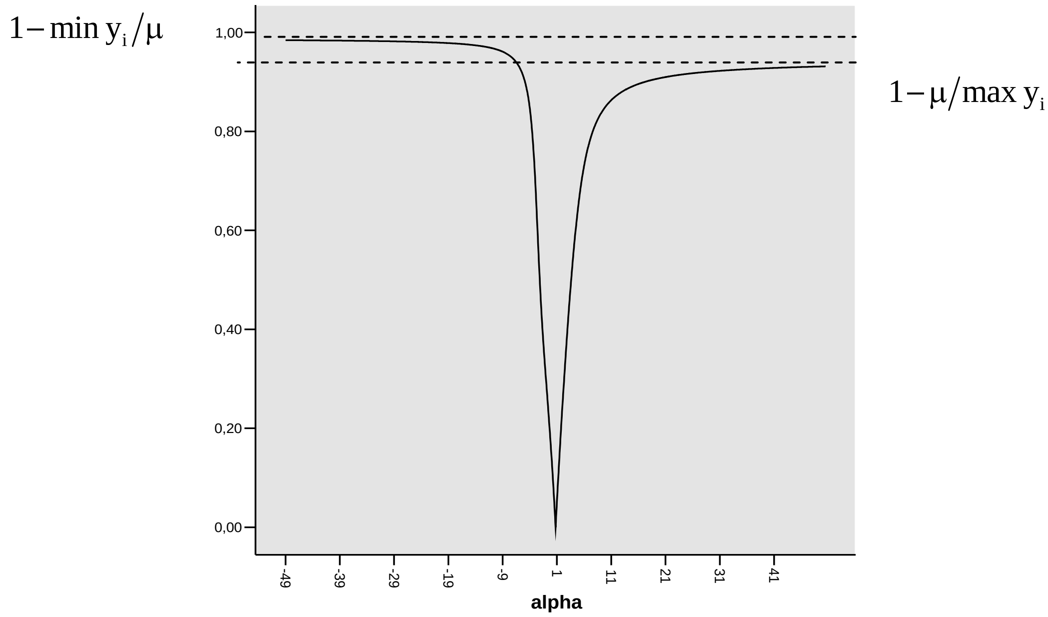

Figure 2. I_α^{A*} -curves for the distributions x and y

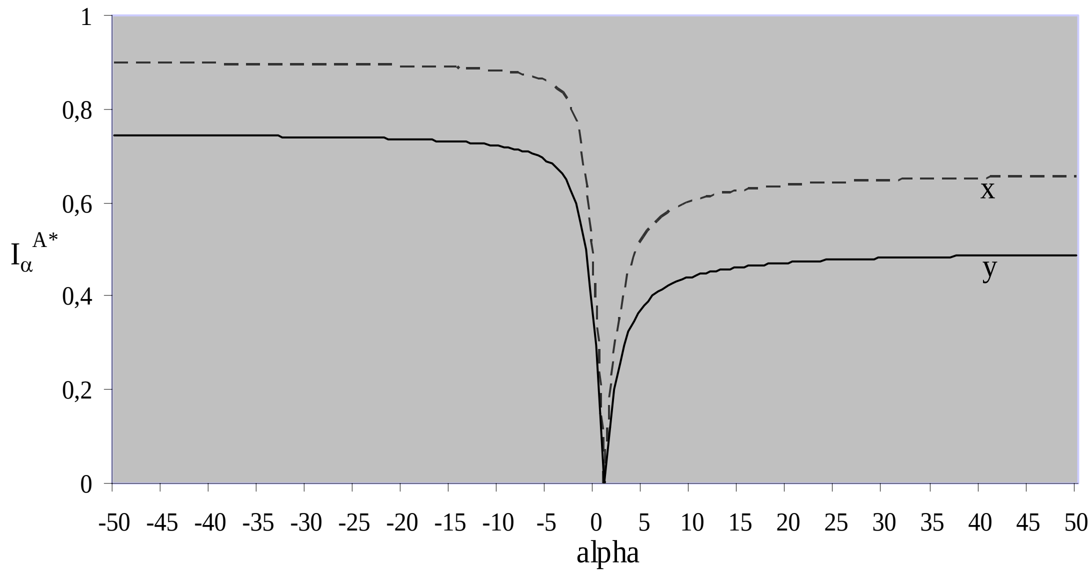

Figure 3. Geometric interpretation of a multiplicatively decomposable measure for a given y