The Employed, the Unemployed, and the Unemployable: Directed Search with Worker Heterogeneity[*]

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Abstract

We examine the implications of worker heterogeneity on the equilibrium matching process, using a directed search model. Worker abilities are selected from a general distribution, subject to some weak regularity requirements, and the firms direct their job offers to workers. We identify conditions under which some fraction of the workforce will be "unemployable": no firm will approach them even though they offer positive surplus. For large markets we derive a simple closed form expression for the equilibrum matching function. This function has constant returns to scale and two new terms, which are functions of the underlying distribution of worker productivities: the percentage of unemployable workers, and a measure of heterogeneity (κ). The equilibrium unemployment rate is increasing in κ and, under certain circumstances, is increasing in the productivity of highly skilled workers, despite endogenous entry. A key empirical prediction of the theory is that $\kappa \geq 1$. We provide evidence that this prediction is supported by data from several countries.

1 Introduction

The distribution of worker productivities is clearly a key factor in determining the outcomes of the labor market, and macroeconomies more generally. In a general sense, it is well known that this distribution can affect the matching process, the unemployment rate, the types of unemployment observed, and the distribution of wages received by workers. Understanding these relationships deeply, though, requires a theory of the labor market which places worker heterogeneity at the heart of the analysis, and which draws out the implications of this heterogeneity as phenomena that occur in equilibrium.

Directed search theory is a natural candidate for this type of analysis because rational selection across a variety of alternatives is its central focus. The location choice, in directed search settings, is made by buyers, who choose across sellers. The key insight of directed search was identified in Peters (1984): buyers, when considering which seller to approach, trade off two factors – the payoff from the good, and the probability of successful purchase. Thus, sellers that offer higher payoffs will typically attract more buyers and a lower probability, from the point of view of buyers, of a successful purchase. When applied to the labor market, the implications of this insight depend on which side of the market is interpreted as buyer and seller. For example, Montgomery (1991), and Burdett, Shi, and Wright (2001), modelled firms as sellers, and workers as buyers of jobs; whereas Julien, Kennes, and King (2000) modelled the market the other way, with workers interpreted as sellers, and firms as buyers, of labor. When focussing on the implications of worker (rather than firm) heterogeneity, arguably, it is more natural to use the latter approach, where firms select over a distribution of worker types.[1]

One of the more attractive features of directed search theories of the labor market is that, in their simplest forms, they produce equilibrium matching processes that have many of the characteristics of the matching functions that are com-

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[1]In the simplest case with heterogeneous workers and homogeneous firms, if workers act as buyers and choose which firms to approach, then heterogeneity plays no role at all in the matching process. See Julien, Kennes, and King (2005) for further details.

monly supported by empirical work: matches are increasing in vacancies and un-employment, concave, and (with large markets) have constant returns to scale (Julien, Kennes, and King (2000)). In models with homogeneous workers of this sort, however, this equilibrium matching process has no parameters in it – a fact that weakens its performance empirically, relative to, for example, Cobb-Douglas matching functions (Petrongolo and Pissarides (2001)). Directed search models do exist now that have parameters in the implied equilibrium matching processes (see, for example, Albrecht *et al* (2004), where the number of applications plays this role), and worker heterogeneity can play a role here, at least in principle. [2]

Some of the implications of heterogeneity in directed search models have already been explored. For example, Shi (2002) considers a model with two types of workers (skilled and unskilled) and two types of firms ("high-tech" and "low-tech") where all workers are equally productive in low-tech jobs but skilled workers outperform unskilled ones in high-tech jobs. The focus, in that paper, is on wage inequality and the responses to skill-biased technological progress. Julien, Kennes, and King (2006a) study a dynamic environment where heterogeneous firms approach homogeneous workers, and characterize the equilibrium wage distribution as a function of the distribution of firm productivities and entry costs. Shi (2001), Shimer (2005), and Eeckhout and Kircher (2010), consider environments with two-sided heterogeneity, and identify conditions under which equilibria exhibit positive assortative matching.

In this paper, we are interested, specifically, in understanding the matching process and the unemployment rate as functions of an existing skill distribution for workers – and how these respond to changes in this distribution – in equilibrium. To isolate these effects, in the simplest environment, we consider a model with only one-sided heterogeneity. We examine the implications of worker heterogeneity using a directed search model based on Julien, Kennes, and King (2000), but with a non-degenerate distribution of worker productivities. We characterize the unique symmetric mixed strategy equilibrium of this model, under different

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[2]Petrongolo and Pissarides (2001) suggest a variety of ways in which parameters could be introduced into urn-ball matching functions, which are the mathematical form that many equilibrium matching processes in directed search models take. Although their analysis is not, strictly speaking, directed search, it is very suggestive about the effects of heterogeneity in directed search. See, also, Stevens (2007), for an alternative approach to providing microfoundations for the matching function.

assumptions about the nature of the distribution of worker types. We start with the simplest case: binomial distributions. We then go on to consider arbitrary distributions with continuous supports. In both cases, we characterize the nature of the equilibrium and study comparative static properties, as the underlying distribution of worker types changes. We also provide simple closed form solutions for the equilibrium matching process, in both cases, where the matching process is a function of statistics of the underlying distribution of worker productivities. The performance of this matching process is then assessed, using data from the United States and other countries.

We find that, for each distribution of worker types, we can identify two different classes of workers: those who will be approached with positive probability, and those who will not. Thus, we identify a set of "unemployable" workers: those who will never be approached by firms, even though they offer positive surplus. The intuition for their existence, in equilibrium, follows directly from the central insight of directed search, mentioned above. When firms choose which workers to approach, they are influenced by both the potential productivity the worker offers and the probability of being able to hire the worker. With arbitrary distributions of worker types, in principle, one can identify cut-off values of worker types (at the lower end) where firms would prefer to take a chance on being able to hire more productive workers, even if they could hire the lower productivity ones with probability one. Thus, the presence of high productivity workers introduces the possibility that some low productivity workers will simply never be offered jobs. Moreover, increasing the productivity of the best workers may, *ceteris paribus*, increase the fraction of workers at the lower end that are unemployable in equilibrium and increase the unemployment rate more generally.

We also identify three conceptually distinct effects that heterogeneity has on the equilibrium matching in the model. The first is the *decreased employability effect*, which arises due to the potential existence of unemployables introduced by heterogeneity. The second is the *increased miscoordination effect*. Minimization of the unemployment rate requires that each firm assign equal probability to approaching each worker. This occurs in the equilibrium with homogeneous workers, but not with heterogeneous ones, where more productive workers are assigned higher probabilities. (However, the equilibrium we consider here is constrained-efficient in the sense that it maximizes expected total surplus.) The third effect is the

decreased effective labor force effect: the probability of being employed is higher for those workers who fall into the category of being "employable", with heterogeneity. However, we show that, overall, heterogeneity increases the equilibrium unemployment rate.

The implied matching function has two terms in it that are functions of the underlying distribution of worker productivities. The first is the fraction of the workforce that is unemployable, q, and the second is a measure of heterogeneity, κ, that we construct. This second measure, according to the theory, must be weakly greater than one (the lower bound, with homogeneous workers). We examine whether or not this holds true in the data by considering two different empirical exercises. In the first exercise, we treat κ as a variable and use the matching function to generate a series $\{\kappa_t\}$. In the second exercise, we treat κ as a parameter and estimate its implied value by estimating the matching function in the way that is standard in the literature. This provides a weighted average value of κ. In both cases, the implied values of κ are greater than one – in accordance with the theory.

The remainder of the paper is organized as follows. Section 2 introduces the model with finite numbers of agents. In Section 3, we consider the case with arbitrarily large numbers of workers and firms, both with a fixed ratio of the two and with endogenous entry and derive most of the theoretical results. Section 4 identifies constrained efficient allocations in the model. Section 5 presents the empirical results. Concluding remarks are given in Section 6.

2 The Model with Finite Numbers of Agents

Consider a model where $M \in \mathbb{N}$ homogenous vacancies choose across $N \in \mathbb{N}$ heterogenous workers, and where each worker $i = 1, 2, ..., N$ has productivity y_i and $y_1 \geq y_2 \geq ... \geq y_N > 0$. Assume that firms choose whether to approach a particular worker after observing the vector of workers productivities.[3] Assume, further, that each firm has only one vacancy and can approach only one worker to fill it. Once firms have chosen which worker to approach, they make their wage

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[3] Thus, as in Julien, Kennes, and King (2000), this is a model with multiple job applications, where each worker applies to all firms, and firms can perfectly observe each worker's productivity level. See Albrecht, Gautier, and Vroman (2006), Galenianos and Kircher (2009), and Kircher (2009) for models where workers apply only to a fraction of firms.

offers to the worker they face. If a firm is alone when it approaches a worker, it is able to make a take it or leave it offer, and capture all the surplus from the relationship. Alternatively, if more than one firm approaches the same worker, then Bertrand competition among the firms results in worker $i's$ wage being equal to her productivity y_i – in which case the firm that hires the worker captures zero surplus.[4]

In a symmetric mixed strategy Nash equilibrium worker i is visited with the same probability π_i by all firms. For a firm, in this equilibrium, the expected payoff from visiting worker i is therefore the probability of being alone when approaching that worker, multiplied by the worker's productivity: $(1-\pi_i)^{M-1}y_i$. If both workers i and j are visited with positive probabilities in the equilibrium then the firms should earn the same expected payoff from visiting the workers, i.e.

$$(1 - \pi_i)^{M-1}y_i = (1 - \pi_j)^{M-1}y_j. \tag{1}$$

Equation (1) implies that, in equilibrium, workers with higher productivities are visited with higher probabilities. Therefore, in general, there exists a number $K \leq N$ such that $\pi_i > 0$ for $i = 1, ..., K$ and $\pi_i = 0$ for $i > K$. That is, all workers $i = K + 1, ..., N$ are *unemployable* in the sense that all firms assign zero probability to visiting them, even though they have positive surplus associated with them. All other workers $i = 1, ..., K$ are employable but, at the end of the

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[4]Julien, Kennes, and King (2006b)) show that this wage determination mechanism is equivalent to the "Mortensen rule" (Mortensen (1982)), and can be thought of as the outcome of an auction through which workers sell their labor, and where they set their reserve wage equal to their outside option. In this static game, the outside option is normalized to zero. Julien, Kennes, and King (2000) show that the Nash equilibrium reserve wage converges to the outside option as the market becomes large.

We chose this mechanism because of its simplicity: the value of the wage rate is easy to find (either zero or y). However, due to the starkness of the induced wage distribution, this may lead one to question the realism of this distribution. As a check, we also considered an alternative mechanism: generalized Nash bargaining, where, after firms were assigned to workers, each worker (with multiple firms assigned) bargains sequentially with the assigned firms. This leads to a more realistic wage distribution (see Camera and Selcuk (2009) for a discussion of similar mechanisms) but is significantly more cumbersome to work with. In this paper, the emphasis is more on the properties of the equilibrium matching process than on the wage distribution *per se*. In large markets (the focus of the analysis in this paper) the matching processes for the two different wage determination mechanisms converge. (Details of this comparison are available upon request.) In subsection 3.3.3, below, we also consider an alternative with wage posting, based on Eeckhout and Kircher (2010), which generates precisely the same equilibrium matching process.

day, will be either employed or unemployed according to the visit probabilities π_i. In general, for worker i, the probability of being unemployed is $(1 - \pi_i)^M$.

The values of the equilibrium probabilities π_i can be found through the following algorithm.

Step 1. Set $K = N$ and use (1) together with the normalization condition

$$\sum_{i=1}^{K} \pi_i = 1 \tag{2}$$

to find

$$\pi_i = \frac{\sum\limits_{j=1}^{K}(y_i/y_j)^{\frac{1}{M-1}} - (K-1)}{\sum\limits_{j=1}^{K}(y_i/y_j)^{\frac{1}{M-1}}}. \tag{3}$$

Step 2. If $\pi_K > 0$ then (3) gives the equilibrium probabilities. Otherwise set $\pi_K = 0$, decrease K by 1, and go to step 1.

Note that this process will eventually halt. Indeed, it is easy to see that for $K = 2$

$$\pi_i = \frac{y_i^{\frac{1}{M-1}}}{y_i^{\frac{1}{M-1}} + y_j^{\frac{1}{M-1}}} > 0, \text{ for } i = \overline{1,2}, \ j \neq i. \tag{4}$$

In particular, formula (4) implies that in any equilibrium at least two workers are visited. Indeed, if all firms visit only the most productive worker, they are guaranteed to earn zero payoffs, while the deviator will earn payoff y_2. Therefore all the firms visiting the same worker cannot be the equilibrium, and we have arrived at the following proposition.

Proposition 1 *The matching game with finite numbers of players possesses a symmetric mixed strategy equilibrium in which the K most productive workers are visited with positive probabilities, while the $(N - K)$ least productive workers are visited with probability zero. The number K satisfies $2 \leq K \leq N$ and can be found together with matching probabilities if one follows steps 1 and 2 described above.*

The expected number of matches is given by:

$$x = \sum_{i=1}^{N}[1 - (1 - \pi_i)^M],\tag{5}$$

where the probabilities π_i are determined by the algorithm given above. This formula for the matching function holds true for any distribution of worker productivies, and any finite numbers of workers and firms. While this formula is useful for some purposes, it is not particularly amenable either to comparative static analysis of the effects of heterogeneity, or to econometric estimation. To find formulas that are more useful for these purposes, we will examine equilibria in the limit of large economies.

Before doing so, though, it is useful to identify the equilibrium entry condition. Given a fixed cost of entry $k > 0$, entry will occur up to the point where

$$(1 - \pi_i)^{M-1}y_i = (1 - \pi_j)^{M-1}y_j = k\tag{6}$$

for all i in which $\pi_i > 0$, as determined by the above algorithm.

3 The Model with Large Numbers of Agents

In this section, we analyze equilibrium allocations in the economy in the limiting case when $M, N \to \infty$ in such a way that their ratio $\phi = M/N$ remains fixed, and characterize the equilibrium matching process as a function of the underlying distribution of worker productivities. To do so, it is instructive to consider different cases, with increasing degrees of generality. We start with the simplest possible case, where there are only two possible values of y.

3.1 Binomial Productivity Distributions

Suppose that productivities of workers come from a binomial distribution over $\{y_H, y_L\}$ with $y_H \geq y_L > 0$ and $\Pr(y = y_H) = p$. We first prove that, given any fixed number M of vacancies, if there are sufficiently many high productivity workers in a symmetric equilibrium, low productivity workers are visited with zero probability. Let q denote the fraction of the workforce that is employable. There

are, therefore, two cases to consider: $q = p$ when only high productivity workers are employable, and $q = 1$ when all are.

Let N_L and N_H be the number of low productivity and high productivity workers, respectively, in the sample. Hence, from the algorithm given above, the probability that a low productivity worker is visited is

$$\pi_L = \max(0, \frac{N_H(y_L/y_H)^{1/(M-1)} - (N_H - 1)}{N_H(y_L/y_H)^{1/(M-1)} + N_L}). \tag{7}$$

Therefore if

$$N_H \geq \frac{1}{1 - (y_L/y_H)^{1/(M-1)}} \tag{8}$$

then $\pi_L = 0$.

Dividing both sides of (8) by N, taking into account that

$$(y_L/y_H)^{1/(M-1)} = 1 + \frac{1}{M}\ln\frac{y_L}{y_H} + o(\frac{1}{M}) \tag{9}$$

and, noting that $\phi = M/N$, one obtains, after some rearrangement, that, asymptotically, $\pi_L = 0$ if and only if

$$p \geq \frac{\phi}{\ln(y_H/y_L)}. \tag{10}$$

In particular, if competition for workers is strong, i.e.

$$\phi > p\ln(y_H/y_L), \tag{11}$$

then $q = 1$ and all workers are approached at the equilibrium.[5]

We consider each case in turn. Assume, first, that condition (10) holds, so that $q = p$ and all low productivity workers are unemployable. The expected number of matches is given by

$$x(N) = \sum_{i=1}^{N_H}[1 - (1 - \pi_H)^M]. \tag{12}$$

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[5]The same result can be obtained by assuming that every firm approaches each high productivity worker with probability $\pi_H = 1/N_H$ and requiring that it is unprofitable to deviate and approach a low productivity worker, i.e. that $(1 - 1/N_H)^{M-1}y_H > y_L$.

Define the matching rate

$$\xi(p, \phi) = p \lim_{N \to \infty} \frac{x(N)}{N}. \tag{13}$$

Then we obtain the equilibrium matching function:

$$\xi(p, \phi) = p(1 - \exp(-\frac{\phi}{p})). \tag{14}$$

This matching function is a straightforward generalization of the urn-ball matching function discussed in Petrongolo and Pissarides (2001), and derived in directed search models with homogeneous populations in Julien, Kennes, and King (2000) and Burdett, Shi, and Wright (2001). Indeed, with a homogenous population ($p = 1$), one obtains the familiar function: $\xi(1, \phi) = 1 - \exp(-\phi)$. Notice that the more general matching function (14), preserves the key properties of the function in the homogeneous case. In particular, in levels, it has constant returns to scale in M and N. We can also see, very clearly, the effects of heterogeneity on the number of matches. First, notice that the matching rate is independent of the values of y_L and y_H, it depends only on the relative numbers of each type of agent, summarized in the value of p. With heterogeneity, $p < 1$, and we have two effects on the probability of employment. Low productivity workers become unemployable and are not visited at all in equilibrium, however the probability of employment conditional on being a high type, $1 - \exp(-\frac{\phi}{p})$, increases relative to the case with a homogenous population. The total effect of heterogeneity is, however, to decrease employment:

$$\frac{\partial \xi(p, \phi)}{\partial p} = 1 - \frac{\frac{\phi}{p} + 1}{\exp(\frac{\phi}{p})} > 0 \tag{15}$$

and, therefore, $\xi(p, \phi)$ is maximized at $p = 1$, where all workers are homogeneous.

Now let us consider the other case, where condition (8), for a fixed population or (10) for the infinite population does not hold. In this case, $q = 1$ and both types of workers are approached with positive probabilities in the equilibrium.

The visitation probabilities are given by:

$$\pi_L = \frac{N_H(y_L/y_H)^{1/(M-1)} - (N_H - 1)}{N_H(y_L/y_H)^{1/(M-1)} + N_L} \tag{16}$$

$$\pi_H = \frac{N_L(y_H/y_L)^{1/(M-1)} - (N_L - 1)}{N_L(y_H/y_L)^{1/(M-1)} + N_H}. \tag{17}$$

In the limit, with large values of M and N, one obtains

$$\pi_L = \frac{1}{N}(1 - \frac{N_H}{M}\ln\frac{y_H}{y_L}) + o(\frac{1}{N}) \tag{18}$$

$$\pi_H = \frac{1}{N}(1 + \frac{N_L}{M}\ln\frac{y_H}{y_L}) + o(\frac{1}{N}). \tag{19}$$

Let

$$\kappa = (1 - p)(\frac{y_H}{y_L})^p + p(\frac{y_L}{y_H})^{1-p}. \tag{20}$$

The matching rate, in this case, is:

$$\xi(\phi, \kappa) = 1 - \kappa\exp(-\phi). \tag{21}$$

Thus, as before, the matching function has constant returns to scale in M and N but, in this case, is not independent of the values of y_L and y_H. The value of κ depends on the relative value y_L/y_H, and the proportion of workers with high productivity: p. It is easy to show that, for any value of p, κ is minimized where $y_L/y_H = 1$, that is, in the homogeneous case – in which case, $\kappa = 1$, and we recover the homogeneous matching rate as the special case:

$$\xi(0, \phi; y_H, y_L) = \xi(1, \phi; y_H, y_L) = \xi(p, \phi; y, y) = 1 - \exp(-\phi). \tag{22}$$

The function κ is a measure of the heterogeneity of the underlying distribution of worker productivities, and is strictly increasing in the ratio y_H/y_L. Thus, it is clear from (21) that heterogeneity reduces the matching rate in equilibrium.

3.1.1 Vacancy Entry

We now consider the properties of the equilibrium when vacancies can enter by paying a fixed cost $k \in (0, y_H)$.[6] Once again, there are two cases to consider: when some workers are unemployable $(q = p)$ and when not $(q = 1)$. We start with the first case.

Entry with Unemployable Workers   Here, from (6), we have:

$$(1 - \pi_H)^{M-1} y_H = k, \tag{23}$$

where $\pi_H = 1/N_H$. In the large market, this becomes:

$$y_H \exp(-\phi/p) = k. \tag{24}$$

Solving this equation for the tightness ratio ϕ, we get:

$$\phi = p \left(\ln y_H - \ln k \right) \tag{25}$$

Now, recall condition (10), which must hold when low productivity workers are unemployable. Re-rewriting this condition, we have:

$$\phi \leq p \left(\ln y_H - \ln y_L \right). \tag{26}$$

The above two conditions imply the following:

$$y_L \leq k. \tag{27}$$

This tells us that low productivity workers will be unemployable if and only their productivity level is below the cost of vacancy creation. We summarize this result in a proposition.

Proposition 2 *In the binomial model with entry, a worker is employable if and only if $y > k$.*

Let us now consider, then, the case where all workers are employable; that is, when $q = 1$.

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[6]Below, in Subsection 3.3.4, we also consider the case where entry costs differ across firms.

Entry without Unemployable Workers $(k < y_L)$ In this case, the entry condition (6) becomes:

$$(1 - \pi_L)^{M-1} y_L = (1 - \pi_H)^{M-1} y_H = k, \tag{28}$$

where π_L and π_H are given in (18) and (19) respectively. From here, using (19), we have

$$y_H \left[1 - \frac{1}{N}(1 + \frac{N_L}{M} \ln \frac{y_H}{y_L}) \right]^{M-1} = k. \tag{29}$$

In the limit, this becomes

$$y_H \exp\left[-\phi - (1-p)\ln(y_H/y_L) \right] = k. \tag{30}$$

Solving for ϕ, we get:

$$\phi = p(\ln y_H - \ln y_L) + (\ln y_L - \ln k). \tag{31}$$

Lemma 1 *In the binomial model with entry, if all workers are employable then, in equilibrium, $\partial\phi/\partial p > 0$, $\partial\phi/\partial y_H > 0$, $\partial\phi/\partial y_L > 0$, $\partial\phi/\partial k < 0$.*

Proof. This follows directly from (31). ∎

3.1.2   The Equilibrium Unemployment Rate

In general, the equilibrium unemployment rate is given by $1 - \xi$. The value of ξ depends on whether or not low productivity workers are unemployable.

When Low Productivity Workers are Unemployable   In this case, $q = p$ and the matching rate ξ is given in (14). In this case, the equilibrium unemployment rate is:

$$U = 1 - p + p \exp(-\phi/p), \tag{32}$$

where ϕ is given in (25). Making the substitution, we have:

$$U = 1 - p + p \left(\frac{k}{y_H} \right). \tag{33}$$

Clearly, from (33), the unemployment rate has two components in this case. The first component, $1-p$, captures the fraction of workers who are unemployable. The second term captures those who are employable (in the sense that each firm assigns positive probability to approaching them), but unemployed due to the fact that, ex post, they were not approached by any firms. Note that, in the homogeneous case, when $p=1$, then $U = k/y_H$, as is standard.

Proposition 3 *In the binomial model, if low productivity workers are unemployable ($q = p$) then equilibrium unemployment is increasing in k and decreasing in y_H and p.*

Proof. Follows directly from (33). ∎

The effects of increases in k and y_H on unemployment are entirely straightforward. Increases in the entry cost k reduces profitability, entry, and, thus, increases unemployment. Increases in productivity y_H have precisely the opposite effect. Increases in p have two effects: they reduce the fraction of the workforce that is unemployable (driving down unemployment), but increase the fraction that are subject to the possibility of coordination unemployment. Since unemployable workers are unemployed with probability one, and employable workers are unemployed with probability less than one, the overall effect is negative.

When Low Productivity Workers are Employable In this case, $q = 1$, and the matching rate is given in (21). The equilibrium unemployment rate is:

$$U = \kappa \exp(-\phi), \tag{34}$$

where $\kappa = \kappa(y_H, y_L, p)$ is given in (20) and $\phi = \phi(y_H, y_L, p, k)$ is given in (31).

Proposition 4 *In the binomial model, if low productivity workers are employable ($q = 1$) then equilibrium unemployment is increasing in k, decreasing in y_L and p. The effect of y_H on unemployment depends on the value of y_H/y_L. When $y_H \approx y_L$, unemployment is decreasing in y_H. Beyond a critical point of y_H/y_L, unemployment is increasing in y_H.*

Proof. From (34), using (20) and (31) we obtain:

$$\frac{\partial \ln U}{\partial k} = \frac{1}{k} > 0 \tag{35}$$

$$\frac{\partial \ln U}{\partial y_L} = -\frac{1-p}{\kappa y_L}(\kappa + p(\frac{y_H}{y_L})^p(1 - \frac{y_L}{y_H})) < 0 \tag{36}$$

$$\frac{\partial \ln U}{\partial p} = -\frac{1}{\kappa}(\frac{y_H}{y_L})^{p-1}(\frac{y_H}{y_L} - 1) < 0 \tag{37}$$

$$\frac{\partial \ln U}{\partial y_H} = \frac{p}{\kappa y_H}((1-p)(\frac{y_H}{y_L})^{p-1}(\frac{y_H}{y_L} - 1) - \kappa). \tag{38}$$

It is straightforward to show that $\partial \ln U/\partial y_H < 0$ when $y_H \approx y_L$. Also, this derivative changes sign when y_H/y_L crosses the value ζ, where ζ is determined as the unique solution to:

$$(1-p)(\zeta^p - \zeta^{p-1}) = \kappa. \tag{39}$$

∎

These results are quite intuitive. The effects of increasing vacancy costs, k, are entirely straightforward. They do not affect heterogeneity (κ) but simply reduce the profits of entry, thus reducing entry, and thereby increasing unemployment, as in the previous case. Increases in the output of low productivity workers y_L have two effects, both of which work in the same direction on unemployment. The first is to increase average productivity, thus encouraging entry and reducing unemployment. Secondly, increases in y_L compress the distribution of worker productivities, reducing the measure of heterogeneity, κ, and thereby increasing the efficiency of the matching process and reducing unemployment. Increases in the fraction of workers with high productivity, p, also has two effects. The first, again, increases average productivity (thereby inducing further entry and reducing unemployment). The second affects the degree of heterogeneity (κ). It is straightforward to show that the function $\kappa(p)$ is inverse U-shaped, with minimum values of 1 at $p = 0$ and $p = 1$. However, the overall effect of any increase in p on unemployment is always negative.

Increases in the output of high productivity workers y_H also have two effects on unemployment. As with increases in y, these increase average worker productivity, and induce entry. However, when y_H increases, it spreads the distribution of productivity, and increases κ, reducing the efficiency of matching. These two

effects move in opposite directions. When heterogeneity is small, the first effect dominates, and unemployment is decreasing in y_H. However, as heterogeneity increases, the second effect dominates the first, and further increases in y_H actually increase unemployment.

We now turn to consider more general skill distributions, where productivities are sampled from an arbitrary distribution with c.d.f. $F(\cdot)$. Since the set of piecewise constant distribution functions, which correspond to random variables with finite support, is everywhere dense in the set of all distribution cases, we start with considering the case when the productivity can take only a finite set of values $\{y_1, ..., y_I\}$ with $\Pr(y = y_i) = p_i$ and then generalize to the case of arbitrary distributions.

3.2   Distributions with Finite Supports

Consider the case when the set of all possible productivities is finite. Without loss of generality assume $y_1 > y_2 > ... > y_I$. Let N_i be the number of workers with productivity y_i and

$$\sum_{j=1}^{I} N_j = N. \tag{40}$$

Formula (3) implies:

$$\pi_i = \frac{\sum_{j=1}^{K} N_j (y_i/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K} N_j + 1}{\sum_{j=1}^{K} (y_i/y_j)^{\frac{1}{M-1}}}, \tag{41}$$

where K is the lowest productive worker visited at equilibrium with positive probability. If

$$\sum_{j=1}^{I} N_j (y_I/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{I} N_j + 1 > 0, \tag{42}$$

then all types are visited with positive probability. Otherwise, K is found from

$$\sum_{j=1}^{K} N_j (y_K/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K} N_j + 1 > 0 \geq \sum_{j=1}^{K+1} N_j (y_{K+1}/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K+1} N_j + 1. \tag{43}$$

Taking the limit as population increases, and proceeding in the same way as above, one obtains:

$$\pi_i = \frac{1}{K}(1 - \sum_{j=1}^{K} \frac{N_j}{M} \ln \frac{y_j}{y_i}) + o(\frac{1}{N}).$$ (44)

In general there exists a critical productivity y^* such that workers are approached with positive probability if $y_i \geq y^*$ and are not approached otherwise. We will determine this productivity shortly. Given y^* for the large population, the fraction of workers who are employable is $1 - F(y^*)$, i.e.

$$p \lim_{N \to \infty} \frac{K}{N} = 1 - F(y^*),$$ (45)

which implies

$$\pi_i = \frac{1}{N}(\frac{1}{1 - F(y^*)} - \frac{E(\ln y | y \geq y_K) - \ln y_i}{\phi}) + o_p(\frac{1}{N}),^7$$ (46)

provided $y_i > y_K$ and $\pi_i = 0$ otherwise.

The critical productivity y^* is determined by:

$$\frac{\phi}{1 - F(y^*)} = E(\ln y | y \geq y^*) - \ln y^*.$$ (47)

3.3 Distributions with Continuous Supports

It is easy to see that formulae (46) and (47) continue to hold if the productivities are selected from an arbitrary distribution $F(\cdot)$ with support $[\underline{y}, \overline{y}]$, where $\infty > \overline{y} \geq \underline{y} \geq 0$. Indeed, given such a distribution there exists a sequence of piecewise constant distributions $\{F_n(\cdot)\}_{n=0}^{\infty}$ such that

$$\lim_{n \to \infty} F_n(y) = F(y),$$ (48)

---

[7]For simplicity of notation, hereafter, we drop $o_p(\frac{1}{N})$ and $o(\frac{1}{N})$ when considering large populations.

for all y in the support of $F(\cdot)$. For every distribution $F_n(\cdot)$ formulae (46) and (47) imply

$$\pi_{in} = \frac{1}{N}\left(\frac{1}{(1 - F_n(y_n^*))} - \frac{E_n(\ln y | y \geq y_{Kn}) - \ln y_i}{\phi}\right), \qquad (49)$$

provided $y_i > y_{Kn}$ and $\pi_i = 0$ otherwise. In the case of an absolutely continuous distribution with compact support , the difference between y_{Kn} and y_n^* disappears in the limit of a large population and the critical productivity y_n^* is determined by:

$$\frac{\phi}{1 - F(y_n^*)} = E_n(\ln y | y \geq y_n^*) - \ln y_n^*. \qquad (50)$$

Here E_n denotes expectation calculated using $F_n(\cdot)$. Taking the limit as $n \to \infty$, one obtains (46) and (47), where expectations are calculated using $F(\cdot)$.

Let us now find the matching function. To start, note that

$$\frac{x}{N} = \sum_{i=1}^{K} \frac{N_i}{N}\left\{1 - \left[1 - \frac{1}{N}\left(\frac{1}{1 - F(y^*)} - \frac{E(\ln y | y \geq y^*) - \ln y_i}{\phi}\right)\right]^{N\phi}\right\}. \qquad (51)$$

Equation (51) can be rewritten as:

$$\frac{x}{N} = \sum_{i=1}^{K} p_i\left(1 - \exp\left(-\frac{\phi}{1 - F(y^*)} + E(\ln y | y \geq y^*) - \ln y_i\right)\right) + \frac{1}{N}o_p\left(\frac{1}{N}\right). \qquad (52)$$

Passing to the probability limit one obtains

$$\xi = (1 - F(y^*))\left(1 - E\left(\frac{1}{y} \Big| y \geq y^*\right)\exp\left(-\frac{\phi}{1 - F(y^*)} + E(\ln y | y \geq y^*)\right)\right), \qquad (53)$$

where, as before, y^* is defined as solution to the equation

$$\frac{\phi}{1 - F(y^*)} = E(\ln y | y \geq y^*) - \ln y^*.[8] \qquad (54)$$

---

[8] It is straightforward to check that, for the case of binary distribution above, formulae (53)-(54) reduce to (14) if condition (10) is satisfied and (21) otherwise.

Using equation (54) one can write (53) in a form

$$\xi = (1 - F(y^*))(1 - y^* E(\frac{1}{y}|y \geq y^*)), \tag{55}$$

which implies

$$\xi \geq 0. \tag{56}$$

It is useful to introduce, again, the notation q for the fraction of workers who are employable. In this case:

$$q = 1 - F(y^*), \tag{57}$$

Also, define:

$$\kappa = E(\frac{1}{y}|y \geq y^*) \exp(E(\ln y|y \geq y^*)). \tag{58}$$

Then the matching rate in (55) can be expressed as:

$$\xi = q(1 - \kappa \exp(-\frac{\phi}{q})). \tag{59}$$

Notice that this matching function, once again, has constant returns to scale in M and N, and collapses down to the familiar function for homogeneous workers. In this case, $q = \kappa = 1$. Outside of this special case, to be able to assess the effects of heterogeneity, we must first assess the magnitude of κ in the general case.

Proposition 5 *For any distribution of skills, $\kappa \geq 1$.*

Proof. Consider two Bernoulli utility functions:

$$u_1 = -\frac{1}{y}, \ u_2 = \ln y, \tag{60}$$

Both utility functions belong to CRRA class with coefficients of relative risk aversion:

$$r(u_1) = 2, \ r(u_2) = 1, \tag{61}$$

therefore the first individual is more risk averse than the second. This implies that for any lottery L

$$CE(u_1, L) \leq CE(u_2, L), \tag{62}$$

where $CE(u, L)$ denotes the certainty equivalent of lottery L for the individual with Bernoulli utility function $u(\cdot)$. Moreover, equality is obtained if and only if lottery L is degenerate (see, Mas-Colell, Whinston, and Green, 1995, Ch.6). Consider the following lottery L^*: draw value y from distribution $F(\cdot)$, if $y \geq y^*$ pay the individual y, otherwise discard y and make a new draw. Then one can write κ as:

$$\kappa = -Eu_1(L^*) * \exp(Eu_2(L^*)). \tag{63}$$

Using the definition of certainty equivalent:

$$Eu(L) = u(CE(u, L)) \tag{64}$$

one can transform (63) into:

$$\kappa = \frac{CE(u_2, L)}{CE(u_1, L)} \geq 1, \tag{65}$$

where the last inequality follows from (62). Moreover, $\kappa = 1$ if and only if all the population of employable workers is homogeneous. ∎

Using this Proposition and (56) one can obtain the following bounds for the matching function:

$$0 \leq \xi \leq q(1 - \exp(-\frac{\phi}{q})) \leq 1 - \exp(-\phi), \tag{66}$$

where the second inequality holds as equality if and only if all the population of employable workers is homogeneous.

Formula (59) allows us to identify three effects of heterogeneity on the matching function in general. The *decreased employability effect* signified by the term q in front of the brackets works to depress total employment, due to the fact that some workers are now not approached at all. The *increased miscoordination effect,* signified by parameter κ, tends to depress employment due to the fact that all firms are more likely to approach the most productive workers. This effect tends to decrease probability of employment conditional on the worker been employable. Finally, the *decreased effective labor force effect* signified by q in the denominator of the exponential factor works to increase the conditional probability of employment conditional on being employable. As we argued above, the net effect of hetero-

geneity on the total matching probability is negative. The effect of heterogeneity on the conditional probability of employment conditional on being employable:

$$1 - \kappa \exp(-\frac{\phi}{q}) \tag{67}$$

is, however, ambiguous. It becomes unambiguously positive if $\kappa = 1$, i.e. if all the population of employable workers is homogeneous, and unambiguously negative if $q = 1$, i.e. the entire population is employable.[9]

3.3.1 Vacancy Entry

We now consider the properties of the equilibrium when vacancies can enter, with cost $k \in (0, \overline{y})$.

Proposition 6 *With entry, $y^* = k$.*

Proof. With finite numbers of workers, for any worker i, entry continues until profits are driven to zero:

$$(1 - \pi_i(y_i))^{M-1} y_i = k, \tag{68}$$

where

$$\pi_i = \frac{1}{N}(\frac{1}{q} - \frac{E(\ln y | y \geq y^*) - \ln y_i}{\phi}). \tag{69}$$

In the limit, this entry condition becomes:

$$\exp\left(-\phi\left(\frac{1}{q} - \frac{E(\ln y | y \geq y^*) - \ln y_i}{\phi}\right)\right) y_i = k. \tag{70}$$

This implies:

$$\exp(-\phi/q) \exp(E(\ln y | y \geq y^*)) = k. \tag{71}$$

Rearranging, and recalling that $q = 1 - F(y^*)$, we have:

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[9]Note that, for the binary distribution of skills considered above, $\kappa = 1$, $q = p$ if (10) is satisfied and $\kappa = (1 - p)(\frac{y_H}{y_L})^p + p(\frac{y_L}{y_H})^{1-p}$, $q = 1$ otherwise. Also, one can extend results of this Section to distributions with unbounded support, since any distribution with unbounded support can be represented as a limit of distributions with compact support.

$$\frac{\phi}{1 - F(y^*)} = (E(\ln y | y \geq y^*)) - \ln k. \tag{72}$$

Comparing (72) with the definition of y^* given in (54), we have: $y^* = k$. ∎

Intuitively, y^* is the expected payoff, in equilibrium, for each firm: the worker with productivity y^* can be hired with probability one. With entry, this value is driven equal to the cost of entry k. This, then, determines the equilibrium value of ϕ, through equation (54). Notice also that, from Proposition 6, we can obtain a simple expression for the probability of finding a match, $m(y)$, for a worker with productivity y : $m(y) = 1 - p \lim_{N \to \infty} (1 - \pi_{in})^{M-1}$, where π_{in} is given in (49). Thus:

$$m(y) = \max(1 - \frac{k}{y}, 0). \tag{73}$$

3.3.2   The Equilibrium Unemployment Rate

From Proposition 6, $y^* = k$. As before, the equilibrium unemployment rate is given by $U = 1 - \xi$. Making the substitution from (59), and rearranging slightly, we have:

$$U = 1 - q + q\kappa \exp(-\phi/q), \tag{74}$$

where q, κ, and ϕ are given in (57), (58), and (54), respectively.

With entry, $y^* = k$, and these three equations then become:

$$q = 1 - F(k) \tag{75}$$

$$\kappa = E(\frac{1}{y} | y \geq k) \exp(E(\ln y | y \geq k)) \tag{76}$$

$$\phi = (1 - F(k)) \left[E(\ln y | y \geq k) - \ln k \right]. \tag{77}$$

Thus, the unemployment rate is purely a function of the cost of entry, k, and the distribution of worker productivities $F(\cdot)$. Examining (74) we can see that, as before, there are two qualitatively different types of unemployment in this model. The first term $(1 - q)$ represents the fraction of the workforce that is unemployable. (Notice that this is increasing in k.) The second term $(q\kappa \exp(-\phi/q))$ represents

the fraction of the workforce that is employable (q) but not employed *ex post*, due to the stochastic nature of the matching process. The size of this term depends not only on the size of q, but also on market tightness (ϕ) and the measure of heterogeneity (κ), all of which are functions of k and $F(\cdot)$.

Substituting (75), (76), and (77) into (74), and collecting terms, gives us:

$$U(k) = F(k) + (1 - F(k)) \, E(\frac{1}{y}|y \geq k)k. \tag{78}$$

Proposition 7 *Equilibrium unemployment increases with vacancy costs k.*

Proof. Notice that

$$(1 - F(k))E(\frac{1}{y}|y \geq k) = \int\limits_{k}^{\overline{y}} \frac{f(y)}{y}dy. \tag{79}$$

Therefore, from (78), we have

$$\frac{\partial U}{\partial k} = f(k) + \int\limits_{k}^{\overline{y}} \frac{f(y)}{y}dy - k\frac{f(k)}{k} = \int\limits_{k}^{\overline{y}} \frac{f(y)}{y}dy > 0, \tag{80}$$

which proves the proposition. ∎

This result is, of course, entirely intuitive but the effects are somewhat more complicated than one might expect. The total effect of increases in k on unemployment is unambiguous, although expression (80) contains terms with different signs. Intuitively, an increase in k reduces the entry of vacancies in the usual way, increasing unemployment. However, with $y^* = k$, an increase in k will increase y^*. This has two effects. First, it increases the fraction of workers that are unemployable (thus increasing unemployment); second, it reduces the heterogeneity among those who are employable – pushing down the unemployment rate for those workers. Overall, however, the first two effects dominate the third.

Our next objective is to investigate how the equilibrium unemployment reacts to changes in the skill distribution. We first show that decreasing the heterogeneity of distribution of skills decreases unemployment, provided all the workers are employable.[10]

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[10]We are grateful to Philipp Kircher for making this observation.

Proposition 8 *Assume skill distributions F and G have the same support $[\underline{y}, \overline{y}]$, F second order stochastically dominates the skills distribution G, and the cost of entry satisfies $k \in (0, \underline{y})$. Under these conditions, the unemployment rate corresponding to distribution F is smaller than the unemployment rate that corresponds to distribution G.*

Proof. Given the distribution of skills, $H(\cdot)$, the unemployment rate is given by:

$$U(H) = \int_{\underline{y}}^{\overline{y}} (1 - m(y)) dH(y) = \int_{\underline{y}}^{\overline{y}} \frac{k}{y} dH(y), \tag{81}$$

where $H \in \{F, G\}$. Convexity of function k/y and definition of the second order stochastic dominance imply $U(F) \leq U(G)$. $\blacksquare$

The above proposition determines the effect of increased heterogeneity of skills on the unemployment rate, but it does not separate the effects due to increases miscoordination and the effects due to entry. It turns out that the effect of a mean preserving spread in risk on the measure of miscoordination, κ, is ambiguous. To get unambiguous results we must provide an appropriate definition of increase in risk for this purpose.[11]

Definition 1 *Let w and z be random variables with non-negative support. We say that z is obtained from w by **a radial increase in risk** if there exists a variable ε such that w and ε and independent and*

$$z = w\varepsilon \tag{82}$$

Note that definition of ε implies that it has non-negative support. To get some intuition concerning the above definition it is useful to re-write formula (82) in logarithms, i.e.

$$\log z = \log w + \log \varepsilon. \tag{83}$$

Formula (83) shows that $\log z$ is obtained from $\log w$ by combination of a shift and

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[11] We are grateful to John Quiggin for suggesting the appropriate definition of "more risky" for this environment.

a mean-preserving spread in risk.[12] Two cases are of a particular interest. If

$$E(\log \varepsilon) = 0 \tag{84}$$

then $\log z$ is obtained by a mean preserving spread in risk from $\log w$. Since by Jensen's inequality

$$E(\log \varepsilon) \leq \log(E\varepsilon), \tag{85}$$

where equality holds if and only if the distribution of ε is degenerate, in that case $Ez \leq Ew$. Another interesting special case is that of mean preserving radial increase in risk, i.e. the case when $E\varepsilon = 1$. In that case

$$E(\log z) \leq E(\log w). \tag{86}$$

The term "increase in risk" is justified by observation that

$$Var(\log z) = Var(\log w) + Var(\log \varepsilon) \geq Var(\log w). \tag{87}$$

Observe, also that

$$Var(z) = Var(w)(E\varepsilon^2) + (Ew^2)Var(\varepsilon), \tag{88}$$

which implies that, for a mean preserving radial increase in risk,

$$Var(z) \geq Var(w). \tag{89}$$

Though the notion of radial increase in risk is different from the second order stochastic dominance, both capture an intuitive notion that one distribution is riskier than the other. Below, we will also assume $\underline{y} = 0$ and $\overline{y} > 0$. We are now ready to formulate the next proposition.

Proposition 9 *Let the random variable y denote the distribution of workers skills and the random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (82) for some random non-degenerate variable ε. Furthermore, assume both random variables y and z have probability densities and the expectations $E(\frac{1}{y})$, $E(\ln y)$, $E(\frac{1}{\varepsilon})$, $E(\ln \varepsilon)$ are finite. Then there exists a $k^* > 0$ such that*

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[12]Note that if y and ε and independent, so are $\log y$ and $\log \varepsilon$.

for any $k < k^*$ a radial increase in risk will lead to increase in miscoordination, i.e. increase in κ.

Proof. Let $F(\cdot)$ be the cdf of random variable y and $H(\cdot)$ be the cdf of random variable z. We will prove that

$$\kappa(H) \geq \kappa(F), \tag{90}$$

where in agreement with equation (65), $\kappa(Q)$ is defined by[13]

$$\kappa(Q) = \frac{CE(u_1, Q)}{CE(u_2, Q)}, \tag{91}$$

for any arbitrary distribution Q and

$$u_1 = \ln y, \quad u_2 = -\frac{1}{y}. \tag{92}$$

First, consider the case $k = 0$.[14] Let $f(\cdot)$ be the probability density of y and $g(\cdot)$ be probability density of ε. Then $h(\cdot)$, the probability density of z, is given by:

$$h(y) = \int_0^\infty \frac{g(t)}{t} f(\frac{y}{t}) dt. \tag{93}$$

To compute $\kappa(H)$ note that

$$\ln CE(u_1, H) = \int_0^\infty \ln y (\int_0^\infty \frac{g(t)}{t} f(\frac{y}{t}) dt) dy. \tag{94}$$

To evaluate the integral (94) let us make a change of variables: $y = zt$, then

$$\ln CE(u_1, H) = \int_0^\infty \ln z f(z) dz + \int_0^\infty \ln t g(t) dt = \ln CE(u_1, F) + \ln CE(u_1, G), \tag{95}$$

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[13]We make the dependence of κ on the distribution explicit, below.

[14]Though this case is not interesting by itself, since $k = 0$ implies $\phi = \infty$ and $\xi = 1$, i.e. full employment, κ is still formally well defined. By continuity argument, we will be able to extend our result for the economically interesting case of small but positive entry costs.

which implies

$$CE(u_1, H) = CE(u_1, F) * CE(u_1, G).$$ (96)

Similar logic implies

$$CE(u_2, H) = CE(u_2, F) * CE(u_2, G)$$ (97)

and therefore

$$\kappa(H) = \kappa(F) * \kappa(G),$$ (98)

where

$$\kappa(G) = \frac{CE(u_1, G)}{CE(u_2, G)} > 1.$$ (99)

The above argument implies that

$$\kappa(H) > \kappa(F)$$ (100)

for $k = 0$. Since equation (62) implies that $\kappa(Q)$ is continuous in k for any random variable that has a density, the inequality will hold for sufficiently small k and the proposition follows. ∎

So far we have studied the effect of a radial increase in risk on the measure of miscoordination, κ. (Note that this effect is independent of the expectation of ε.) To study its effect on unemployment we must also study how it affects ϕ. This is summarized in the following Lemma.

Lemma 2 *Let the random variable y denote the distribution of worker productivities and random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (82) for some random non-degenerate variable ε with*

$$E \ln \varepsilon < 0.$$ (101)

Furthermore, assume both random variables y and z have probability densities and the expectations $E(\ln y)$ and $E(\ln z)$ are finite. Then there exists a $k^{**} > 0$ such that for all $k < k^{**}$ radial increase in risk will lead to decrease in the equilibrium firm to worker ratio ϕ.[15]

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[15] Note that condition (101) will be satisfied for a mean-preserving radial increase in risk.

Proof. Formula (83) applied to the pair of random variables z and y implies that $E \ln z < E \ln y$. Since

$$\omega(k) = E(\ln w | w > k) \tag{102}$$

is continuous in k for any random variable w with a density and

$$\omega(0) = E(\ln w) \tag{103}$$

is finite for $w \in \{y, z\}$ there exists $k^{**} > 0$ such that for all $k < k^{**}$

$$E(\ln z | z > k) > E(\ln y | y > k) \tag{104}$$

and the Proposition follows from (77). ■

Notice that, when all workers are employable: $k \leq \underline{y}$, $F(k) = 0$, $q = 1$, and

$$\kappa = E_F(\frac{1}{y}) \exp(E_F(\ln y)) \tag{105}$$

$$\phi = E_F(\ln y) - \ln k \tag{106}$$

$$U = E_F(\frac{1}{y})k, \tag{107}$$

where we made explicit the dependence of expectation on distribution F. Let G be obtained from F by a mean preserving spread of risk. Then

$$E_G(\frac{1}{y}) > E_F(\frac{1}{y}) \text{ and } E_G(\ln y) > E_F(\ln y). \tag{108}$$

Thus, such a transformation increases our measures of heterogeneity, entry, and unemployment. The general effect of an increase in heterogeneity on unemployment is summarized in the next Proposition.

Proposition 10 *Let random variable y denote distribution of workers skills and random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (82) for some random non-degenerate variable ε with*

$$E \ln \varepsilon < 0. \tag{109}$$

Furthermore, assume both random variables y and z have probability densities and the expectations $E(\frac{1}{y})$, $E(\ln y)$, $E(\frac{1}{\varepsilon})$, $E(\ln \varepsilon)$ are finite   Then there exists $\overline{k} > 0$ such that for all $k < \overline{k}$ a radial increase in risk will lead to increase in unemployment.

Proof. Let us choose k^*, k^{**} from the proofs of Propositions 8 and 9 respectively, set $\overline{k} = \min(k^*, k^{**})$. Then the Proposition follows from (77). ∎

In particular, the Proposition implies that an average productivity preserving radial increase in heterogeneity of workers' skills will lead to an increase in unemployment for sufficiently low entry costs.

Corollary 1 *Under the assumptions of Proposition 9 there exists ε such that $E\varepsilon > 1$ but unemployment increases for sufficiently small k. That is, it is possible for increases in average productivity to increase equilibrium unemployment.*

Proof. It is sufficient to provide an example. Consider $\ln \varepsilon \sim N(-1, 2)$. Then $E \ln \varepsilon = -1$ and $E\varepsilon = e > 1$. The result the follows from Proposition 10. ∎

Thus, as in the binomial case considered above, circumstances exist when increases in the average productivity of workers may increase the equilibrium unemployment rate, despite the implied increase in vacancy entry, due to the increase in the visit probabilities placed on highly productive workers and the consequent reduction in the visit probabilities of less productive workers. However, at this point, it is worthwhile to stress the efficiency of these equilibria.[16]

3.3.3 A Comparison with Eeckhout and Kircher's Model

At this point, as an aside, it is interesting to compare this model with a version of the model presented in Eeckhout and Kircher (2010).[17] In their model, Eeckhout and Kircher follow the tradition established by Moen (1997) whereby workers and firms meet in "markets", which are spatially distinct. The economy has a continuum of these markets, along with continua of both workers and firms, both of which are heterogeneous. Here, we consider a special case of their model where

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[16] Notice that we could interpret a radial increase in the risk of the worker productivity distribution, of this sort, as representing a form of skill-biased technological change. Thus, under these circumstances, skill-biased technological change increases the equilibrium unemployment rate.

[17] We are grateful to Philipp Kircher for suggesting this comparison.

workers are heterogeneous but firms are not. As in their model, though, within each *market* all workers are homogeneous. Workers post a wage, firms observe the wage and decide to enter a market for the workers of type y. We will refer to this as a type y market. Let $\lambda(y)$ be the equilibrium number of firms per worker of productivity y. In Eeckhout and Kircher, within each market, fimrs are assigned to workers via a matching technology $m(\lambda)$. For example, if one assumes that each firm approaches all workers of the same productivity with the same probability, then in the limit of large markets, the matching function for the workers of a particular productivity is:[18]

$$m(\lambda) = 1 - \exp(-\lambda) \tag{110}$$

and the probability that a firm will match a worker of a particular productivity is $m(\lambda)/\lambda$. In equilibrium firms should be indifferent about entering any market of type y, therefore

$$\frac{m(\lambda)}{\lambda}(y - w) = \eta, \tag{111}$$

where η denotes equilibrium market profits. Since the probability that the firm will match with a worker is strictly decreasing in λ, equation (111) establishes a one to one mapping from w to λ, therefore the workers can be seen as implicitly choosing a value of λ, i.e. the workers problem is to choose w to maximize:

$$\max_{w} m(\lambda)w = \max_{\lambda}(m(\lambda)y - \eta\lambda), \tag{112}$$

which leads to the first order condition

$$m'(\lambda) = \frac{\eta}{y}. \tag{113}$$

If the wage is constrained to be non-negative, which will be the case if workers can guarantee themselves zero utility by being unemployed, a fraction $1 - q = F(\eta)$ of workers will be unemployable. If we allow for free entry then $\eta = k$, where k is the cost of entry. The equilibrium number of firms per worker of type y is given

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[18]Eeckhout and Kircher leave this matching technology as general but, in the spirit of directed search, we use this particular technology.

by:

$$\lambda(y) = \ln\frac{y}{k}, \tag{114}$$

the matching probability for the worker of type y is

$$1 - \frac{k}{y}, \tag{115}$$

which is identical to (73) and the wage she commands is:

$$w = y - \frac{k\ln\frac{y}{k}}{1 - \frac{k}{y}}. \tag{116}$$

Note that with this matching technology there is no wage dispersion conditional on type and that the wage tends to zero as y approaches k. The aggregate matching function ξ can be written as

$$\xi = qE(m(\lambda(y))|y \geq k) = q(1 - \kappa_p \exp(-\frac{\phi}{q})), \tag{117}$$

where $\phi = \overline{\lambda}q$ is the number of firms per potential worker,

$$\overline{\lambda} = E(\ln y|y \geq k) - \ln k \tag{118}$$

is the average number of firms per employable worker and

$$\kappa_p = E(\exp(\overline{\lambda} - \lambda(y)|y \geq k)) = E(\ln y|y \geq k)E(1/y|y \geq k)) = \kappa. \tag{119}$$

Thus, the matching function derived here is robust to these alternative assumptions with respect to the wage formation process.[19] Note, however, that wage distribution will be different in two models. With *ex post* Bertrand pricing, wage formation produces wage dispersion conditional on productivity, while wage posting does not. The distribution of wages in our model is a mean preserving spread in risk of the wage distribution in a wage posting model. [20]

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[19]A crucial assumption is that the workers are risk neutral. If, alternatively, workers are risk averse, the matching function with the Bertrand wage determination remains the same but, in the wage posting model the posted wage falls, facilitating entry, increasing employment, and aggregate output.

[20]In a dynamic version of the model, where risk-averse workers of different abilities decide to

3.3.4 Introducing Heterogeneous Entry Costs

Suppose, alternatively, that entry costs are distributed according to some atomless distribution $G(\cdot)$. Let $N\widehat{\phi}$ denote the potential number of firms on the market. If a fraction b of firms enter the market at equilibrium then the entry cost of the least efficient firm that enters will be $k(\phi)$, where $\phi = b\widehat{\phi}$. The function $k(\cdot)$ is increasing (strictly if $\phi < \widehat{\phi}$), and equilibrium entry is given by:[21]

$$k(\phi) = y^*(\phi) \Rightarrow \phi^*. \tag{120}$$

Let $k^* = k(\phi^*)$. A worker is unemployable if $y < k^*$. Note, however, that almost all firms in the market have entry costs below k^*, which means that, in equilibrium, there are some workers, which have positive productivity, net of entry costs for at least at some firms, but who are still unemployable.

4 Constrained Efficiency

In this Section we identify the allocations that are constrained efficient, for given M and N, and compare them to the equilibrium matching allocations found in Section 2. First, notice that unconstrained efficiency requires that the M most productive workers should be matched to the firms. In particular, if $M > N$ ($\phi > 1$) unconstrained efficiency requires zero unemployment rate. Such matching can be easily achieved if the firms are allowed to move sequentially. If firms move simultaneously, however, such an allocation is not feasible. We will employ the following notion of constrained efficiency. Assume each firms send a request for a worker to a clearing house. The clearing house sends the request to worker i with probability π_i.

We assume that the clearing house cannot keep track of the workers who already received requests. Therefore, the only control variables the clearing house has are probabilities π_i, which are selected to maximize the social surplus of the matches,

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invest in human capital prior to entering the job market, these different wage formation processes can result in different distribution of skills, and affect equilibrium unemployment.

[21]Note that, from (54), y^* is decreasing in ϕ, which implies that ϕ^* is unique.

i.e. the clearing house solves:

$$\max_{\pi} \sum_{i=1}^{N}(1 - (1 - \pi_i)^M)y_i$$
$$s.t. \sum_{i=1}^{N} \pi_i = 1, \pi_i \geq 0 \qquad , \qquad (121)$$

where $\pi = (\pi_1, ..., \pi_N)$ is the vector of probabilities.

Definition 2 *Matching probabilities are constrained efficient if they solve problem (121).*

The Lagrangian for this problem is:

$$L = \sum_{i=1}^{N}(1 - (1 - \pi_i)^M)y_i - \lambda(\sum_{i=1}^{N} \pi_i - 1) + \sum_{i=1}^{N} \mu_i \pi_i. \qquad (122)$$

The Kunh-Tucker conditions are

$$\begin{cases} M(1 - \pi_i)^{M-1}y_i - \lambda + \mu_i = 0 \\ \sum_{i=1}^{N} \pi_i = 1, \ \pi_i \geq 0, \ \mu_i \geq 0 \\ \mu_i \pi_i = 0 \end{cases} \qquad . \qquad (123)$$

To analyze system (123) notice that

$$\frac{\partial^2 L}{\partial y_i \partial \pi_i} = M(1 - \pi_i)^{M-1} > 0, \qquad (124)$$

therefore efficient probabilities are increasing in productivities. Hence, there exists a $K \leq N$ such that $\pi_i > 0$ for $i = 1, ..., K$ and $\pi_i = 0$ for $i > K$.

Lagrange multiplies are given by:

$$\begin{cases} \lambda = M(1 - \pi_K)^{M-1}y_K \\ \mu_i = 0 \text{ for } i = \overline{1, K} \\ \mu_i = M[(1 - \pi_K)^{M-1}y_K - y_i] > 0 \text{ for } i = \overline{K+1, N} \end{cases} \qquad . \qquad (125)$$

Notice that the constraint efficient probabilities coincide with the equilibrium

matching probabilities. That is, equilibrium matching is constrained efficient. Constrained efficiency holds true even with finite numbers of agents, due to the particular wage determination mechanism we use in this paper. It remains true in the limiting large economy, and with entry.[22]

5   Empirical Evidence

In this section we take some of the implications of our theory directly to the data. The theory provides us with the following specification for a matching function, equation (59):

$$\xi = q \left(1 - \kappa \exp\left(-\frac{\phi}{q} \right) \right)$$

where q represents the per cent of employable workers, κ is the measure of heterogeneity, and ϕ is the vacancy-unemployment rate.

A simple test of the theory is to solve (59) for κ to obtain

$$\kappa_t = \exp(\frac{\phi_t}{q_t})\frac{q_t - \xi_t}{q_t} \tag{126}$$

and to use time series data to compare each implied κ_t with one.

Alternatively, one can also estimate the model, treating κ as a parameter. Assuming that κ is constant requires that we assume that the distribution of skills does not change over time. However, the exercise is useful, since it allows us to construct a confidence interval for an unbiased weighted average value of $\{\kappa_t\}$ and test formally that this is greater than one, as the theory predicts. [23]

5.1   Treating κ as a Variable

The theory suggests that κ is a function of the distribution of worker productivity and costs of vacancy creation and potentially time varying. If we take the long-term unemployment to unemployment ratio as the ratio of the unemployable workers in the economy it is possible to use equation (126) to generate the implied

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[22]These last two points are very standard in directed search models, and the proof is omitted here. Details are available from the authors upon request.

[23]This may be useful whenever the quality of individual data is poor and individual observation of κ_t are to noisy for any useful comparisons with theory to be made.

value of κ given the observed data. Doing so gives us an idea of how κ_t varies over time. We conduct this exercise for the US, using the available data from 2001Q1-2007Q1. Our data comes from a variety of sources: Shimer (2007) uses the Current Population Survey to extract an estimate for the rate at which workers transition from unemployment to employment adjusted for time aggregation.[24] The number of vacancies is taken from the JOLTS data set and unemployment data is available from the Bureau of Labor Statistics. Figure 1 illustrates the series $\{\kappa_t\}$ generated from this exercise. Clearly, although there is significant variation in κ over time we find that it exceeds a value of one in all periods, in accordance with the theory.

5.2 Treating κ as Parameter

We begin with an examination of the matching function in the United States. The model in this paper is static, but when estimating a directed search matching function it is necessary to take a stand on the length of a time period during which firms can attract workers. For the purposes of our estimation, we set a period equal to one month. That is, firms are allowed to approach one worker on a monthly basis.[25] Given the instantaneous rate of matching derived by Shimer (2007), we are able to extract the probability that a worker forms a job match within a particular month.

With our available data, we are able to estimate the parameters of our matching function using non-linear least squares. We take the ratio of long term unemployed as a proxy for size of the unemployable population $1 - q$, and allow this to vary over time, according to the data.[26] There are potentially two sources of error in this procedure. Some of the long-term unemployed workers may actually be employable and some of the short-term unemployed may be unemployable.[27] The

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[24]This data was constructed by Robert Shimer. For additional details, please see Shimer (2007) and his webpage http://home.uchicago.edu/ shimer/data/flows/.

[25]Selecting a shorter time period will lower the probability of finding a job within a period. In the data, this lower probability of job finding is equivalent to an increase in frictions and will tend to increase our estimates of κ providing stronger support for a model with heterogeneity relative to a model with worker homogeneity. We do not see selecting a longer time period than a month as reasonable. In particular, we expect firms to be able to send out offers at a rate of at least one per month.

[26]For long-term unemployed we take workers who have been unemployed for six months or longer. For European countries we take workers who have been unemployed for a year or longer as a measure of unemployable. These differences reflect data constraints.

[27]It is difficult to know how important these biases are. We do know that given the job finding

calibration of q seems a reasonable activity to undertake for a couple of reasons. First, it may reduce the standard errors associated with our variable of prime interest, κ. Second, our model does not restrict q to be constant over time.

Thus, we calibrate q_t using unemployment by duration data. Then κ is estimated as the solution to the following problem:

$$\min_{\kappa} \sum_{t=0}^{T} \left(\xi_t - q_t \left(1 - \kappa \exp\left(-\frac{\phi_t}{q_t} \right) \right) \right)^2.$$

It is straightforward to show that the estimate, $\widehat{\kappa}$ will satisfy:

$$\widehat{\kappa} = \frac{\displaystyle\sum_{t=0}^{T} q_t^2 \kappa_t}{\displaystyle\sum_{t=0}^{T} q_t^2}, \tag{127}$$

where κ_t is the true value of time series κ at period t, given by (126). Our theory predicts that $\widehat{\kappa} > 1$ The results are provided in Table 1 and confirm that prediction. This result is strongly statistically significant and supportive of the presence of worker heterogeneity:

Table 1: Estimation of κ

Parameter	κ
Estimate	1.26
Std Error	0.03

5.3   Cross-country Evidence

The above results outline the evidence from the United States. It is possible to examine other countries. Shimer (2007) and Elsby, Hobijn and Sahin (2008) illustrate how to use data on unemployment duration to infer job finding probabilities. We collected data on unemployment by duration and job vacancies for a set of

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rates observed in the data, the probability that a worker with the average rate of job finding remains unemployed after six months is about 10 per cent.

European countries from the Eurostat Database. This allowed us to extract estimates of the matching rate for the following European countries: Finland, Greece, the Netherlands, Portugal, Slovenia and the United Kingdom.[28]

Table 2: Estimation of κ

Parameter	κ
Finland	1.29 (0.04)
Greece	1.19 (0.04)
Netherlands	2.27 (0.23)
Portugal	1.01 (0.01)
Slovenia	1.21 (0.02)
United Kingdom	2.59 (0.09)

We examine the case where the ratio of short-term to total unemployment is used as a measure of employability. The results are provided in Table 2 with standard errors provided in brackets. Under this specification the cross-country data suggests that $\kappa > 1$ with the exception of Portugal where the implied value of κ is consistent with a directed search model of the labour market, but without heterogeneity in worker productivity.

6 Conclusions and Extensions

In this study we have explored the implications of worker heterogeneity on matching and unemployment when firms direct their search. We have used the simplest model possible in the sense that the environment is static, heterogeneity is one-dimensional, and firms are homogeneous. However, we have allowed for considerable generality with regard to the distribution of worker productivities. By doing so, we have identified conditions under which some proportion of the workforce will not be approached in equilibrium. Plausibly, these workers, which we call "unemployables" correspond, in the real world to long-term unemployed workers. They are qualitatively different from workers who will be approached with positive probability, but are unlucky *ex post*, and, thus, simply unemployed. In equilibrium, unemployment is a prospect facing workers of all levels of skill –

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[28] Our choice of countries is generally restricted by the necessity of having vacancy data and unemployment by duration data for a number of years. Spain was removed from the sample due to the fact that the definition of a vacancy changes in mid-sample.

but only those workers at the lower end of the distribution are unemployable, and have no realistic hope of being employed.

We also found that a simple equilibrium matching function emerges in this environment, which is a generalization of the matching function that comes out of directed search models with homogeneity, but has two extra terms: the fraction of unemployable workers and a statistic, κ, that measures the degree of heterogeneity in the distribution. This allowed us to consider the effects of this heterogeneity on the equilibrium matching rate and unemployment, in a relatively straightforward way. In particular, we found that, under certain conditions, an increasing the productivity of high-skilled workers will increase the equilibrium unemployment rate – despite the fact that it encourages entry by firms. Finally, we tested one of the key predictions of the theory ($\kappa \geq 1$) and found support for this in the data.

Several immediate extensions come to mind, which we find interesting. First, in a dynamic version of this model, with aggregate shocks, one might expect that the fraction of workers who are unemployable would vary over the cycle – and, potentially, with more volatility than the fraction of workers who are simply not lucky enough to find work. Secondly, introducing firm heterogeneity alongside worker heterogeneity seems worthy of study. Finally, allowing for investments on either side of the market, and thus, potentially endogenous heterogeneity clearly would be of interest.

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Figure 1