Worth and willingness to pay: Ranking distributions of monotone attributes[1]

Carmen Herrero

University of Alicante & Ivie

Antonio Villar

Pablo de Olavide University & European University Institute

Abstract: This paper refers to the ranking of densities that describe the distribution of an attribute in a given set of populations. The key elements of the problem are: (i) The distributions refer to ordered categorical data (e.g. health statuses, educational achievements, prestige positions, satisfaction levels); (ii) The evaluation of each distribution is relative to the others with which it is compared. We propose an evaluation method that is cardinal, complete and transitive, which based on the consistent application of the "willingness to pay" principle and the likelihood of getting better results when making a random extraction. A characterization of this method, in terms of simple properties, is provided. We show that this evaluation procedure induces an new way of ranking income distributions.

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1 Introduction

Ranking distributions is one of the most basic problems when comparing populations with respect to some attribute. We analyse here the case of distributions regarding variables that measure attributes with the property that higher values are preferable. We can think of variables related to aspects such as health, wellbeing, satisfaction, success, effectiveness, durability, precision, endurance, etc. We call this type of variables **monotone**.

We propose a complete, transitive and cardinal measure that permits one to evaluate the relative desirability of those distributions in terms of the likelihood of getting better results. This measure is based on the application of the notion of *willingness to pay*, a standard criterion used in economics to evaluate those commodities for which there is not a well defined market price (either because they are singular or because there is not a regular market for them).

1.1 Motivation and informal description

Let us motivate the approach in terms of a very simple example. Suppose we have a collection of urns, containing prizes of different categories, and that we have to make an extraction of one of those urns. Assuming that the distribution of prizes within each urn is known, which urn to choose? To make it more precise, take the simplest case in which we have just two urns. The distribution of prizes by categories within each urn is described in Table 1, where categories are ordered from best (category I) to worst (category IV):

Table 1

Prize Category	I	II	III	IV
Shares urn A	0.2	0.4	0.3	0.1
Shares urn B	0.3	0.3	0.2	0.2

We can consider here two different questions. First, which urn is best? Second, how much better is one urn than the other? The first question involves an ordinal assessment (ranking the urns) whereas the second calls for a cardinal evaluation. The ordinal nature of the information makes those questions more challenging.

We propose to address those problems as follows. Let p_{AB} stand for the probability that a random extraction from *A* yields a prize of a higher category than another random extraction from *B*; call this number the **domination probability** of A over *B*. And let p_{BA} be the probability of the opposite (that an extraction from *B* yields a prize of a higher category than one from *A*). Then, if p_{AB} is larger than p_{BA} we declare that urn *A* is better than urn *B*. This is a simple principle, with a clear meaning, which allows comparing any pair of alternatives when there is no information about the precise amount of the prizes. The ratio of the two domination probabilities (or, alternatively, its difference) contains also information on how much better is one than the other. In the example given in Table 1 the domination probabilities are given by: $p_{AB} = 0.36$, $p_{BA} = 0.38$. Therefore, urn *B* turns out to be preferable to urn *A*. Moreover, we can say that *A* is slightly better than *B* (a 5.5% better, to be precise).

Is that the answer we were looking for the two questions above? Not yet. The reason why this way of comparing the desirability of the distributions is not fully satisfactory is because it is a non-transitive criterion. That is, dealing with three or more urns may yield a cycle and thus the impossibility of ranking them, even though we may know the relative evaluation of any pair of urns. The following example shows the problem.

Consider now the case of three urns, *C*, *D*, *E*, whose prize distributions are described in Table 2.

Table 2

Prize Category	I	II	III	IV
Shares urn C	0.5	0	0	0.5
Shares urn D	0	0.7	0.3	0
Shares urn E	0	0.3	0.7	0

Computing the corresponding domination probabilities yields the following results: $p_{CD} = p_{DC} = 1/2$ and $p_{CE} = p_{EC} = 1/2$. So *D* is indifferent to *C* and *C* is indifferent to *E*. Yet *D* turns out to be strictly better than *E*, as $p_{DE} = 0.49$ and $p_{ED} = 0.09$.

In order to keep the basic intuition about the key role of domination probabilities and avoid the intransitivity problem, we slightly reformulate the evaluation criterion by introducing the notions of *opportunity advantage* and *opportunity cost*. Consider again the case of two urns, *A* and *B*, and suppose that making an extraction requires paying a fee q^A for urn *A* and q^B for urn B. The opportunity advantage of choosing from urn *A*, rather than from urn *B*, is given by the product $p_{AB}q^B$: the probability of beating the result of urn *B* times the cost involved in choosing from that urn, which measures what we save by making an extraction from A. The opportunity cost derived from making an extraction from A rather than from B is given by $p_{BA}q^A$: the probability that an extraction from urn B gives a better outcome times the fee paid to choose from A. Consequently, when $p_{AB}q^B > p_{BA}q^A$ it is better to choose urn A rather than urn B, and vice-versa.

It is then natural to evaluate the relative desirability of urn A as the maximum fee that one would be willing to pay in order to choose from this urn rather than from urn B at a fee q^B. This is, precisely, the **willingness to pay** criterion, which associates to urn A the number that balances A's opportunity advantage and A's opportunity cost. That is, given q^B, the evaluation of A is given by the number w_A that solves the equation: $p_{BA}w_A = p_{AB}q^B$. Note that the quotient $w_A / q^B = p_{AB} / p_{BA}$ measures the relative desirability of urn A with respect to urn B and keeps the domination probabilities as the key element in the evaluation. Indeed, in this case the domination probabilities determine the evaluation except for the choice of units. Needless to say, the higher the willingness to pay the more desirable is the urn.

Observe that the extension of this criterion to the case of more than two urns is not trivial, because it is only natural requiring a consistent evaluation. That is, that all distributions be evaluated by the same principle simultaneously. Think of the case of three urns, as in Table 2, for the sake of illustration. To calculate the willingness to pay for urn C, w_C, one should take as the appropriate evaluations for D and E their corresponding willingness to pay, w_D, w_E. But that in turn requires knowing w_C. In other words, applying consistently this principle to the case of three or more urns requires finding a fixpoint of the mapping that associates to each urn its willingness to pay. That is, solving the following system:

$$\left.\begin{aligned} w_C\left(p_{DC}+p_{EC}\right) &= w_D p_{CD} + w_E p_{CE} \\ w_D\left(p_{CD}+p_{ED}\right) &= w_C p_{DC} + w_E p_{DE} \\ w_E\left(p_{CE}+p_{DE}\right) &= w_C p_{EC} + w_D p_{ED} \end{aligned}\right\}$$

Computing the willingness to pay for the three urns in Table 2 yields the following values (normalizing those values so that the mean value equals one): $w_C = 1$, $w_D = 1.37$, $w_E = 0.63$. No indifference appears between any pair of urns and the evaluation fits much better our intuition about the relative desirability of those urns (notice the symmetry of the evaluations). The reason why these results are different from those obtained when using the direct domination probabilities is that now the evaluation of each urn takes into account all *direct and indirect* domination relationships.

1.2 References to the literature and outline of the paper

The notion of domination probabilities, used to compare pairs of distributions of qualitative variables, appears in Lieberson (1976). The concept of *worth* was introduced in Herrero & Villar (2013) as an extension of this notion to the case of g populations. We show here that this concept corresponds to the consistent application of the willingness to pay criterion, which allows enhance the scope of their analysis to a more general scenario (Section 3.1). We also provide here a slightly different proof of the existence of the worth vector, which makes the paper self-contained, and an immediate extension of the model to the continuous case. An easy characterization of the evaluation function is presented in Section 3.2, which helps understanding the relationship between domination probabilities and willingness to pay. A discussion on the extent of this evaluation formula closes the paper in Section 4.

Related evaluation criteria appear in a variety of problems, such as the statistical measure of distributional similarities (Li, Yi & Jestes (2009), Martínez-Mekler *et al* (2009), Gonzalez-Diaz, Hendrichx & Lohmann (2013)), the ranking of income distributions in different contexts (Shorrocks (1983), Bellú & Liberati (2005), Bourguignon, Ferreira & Leite (2007), Yalonetzky (2012), Sheriff & Maguire (2013), Cuhadaroglu (2013)), the analysis of segregation (Reardon & Firebaugh (2002), Grannis (2002)), the evaluation of scientific influence (Pinski & Narin (1976), Laband & Piette (1994), Palacios-Huerta & Volij (2004), Crespo, Li & Ruiz-Castillo (2013)), the comparison of network structures (Rosvall & Bergstrom (2007)), or the allocation of scores in tournaments (Laslier (1997), Slutzki & Volij (2006)).

2 Model and results

2.1 The model

Our problem consists of comparing the distribution of a given variable in g different populations, $G = \{1, 2, ..., g\}$, with $g \geq 2$. The variable may be either discrete or continuous and is to be interpreted as describing an attribute. Let **C** stand for the range of the variable under study. When the variable is discrete, we can think of **C** as a set of *s* ordered categories, $\mathbf{C} = \{c_1, c_2, ..., c_s\}$; when the variable is continuous we can think of **C** as an interval, $\mathbf{C} = [c_{\min}, c^{\max}]$.

Let f_i, $i = 1, 2, ..., g$, be the "density" (with inverted commas) function that describes the distribution of the variable in population *i*. We shall refer to the points of **C** as *levels* and assume that they are linearly ordered with respect to some desirability criterion; that is $c_i > c_j$ implies that level c_i is better than level c_j. When the variable is discrete the density is just the vector of relative frequencies.

A **problem** refers to the comparison of the set of "densities", *F*, that describe the distribution of the variable of reference in the different populations. That is, $F = \{f_1, f_2, ..., f_g\}$. An **evaluation function** is a mapping φ that associates to each problem *F* a vector $\varphi(F)$ of g components (real numbers) that tells us about the relative situation of those populations.

The key ingredient for the evaluation is the probability p_{ij} that an individual from population *i* belongs to a higher level than an individual from population *j*. When the variable is discrete, we can compute this probability as follows:

$$p_{ij} = a_{i1}\left(a_{j2} + a_{j3} + ... + a_{js}\right) + a_{i2}\left(a_{j3} + ... + a_{js}\right) + ... + a_{i(s-1)}a_{js} \qquad [1a]$$

where a_{ir} denotes the relative frequency of value *r* in population *i*.

When the variable is continuous, we will have:

$$p_{ij} = \int_{t=c_{\min}}^{c^{\max}} \int_{x \leq t} f_i f_j dt dx \qquad [1b]$$

We shall assume, for the sake of simplicity in exposition, that $\sum_{j \neq i} p_{ji} > 0$. That is, each population has a positive probability of domination over some other.

The **opportunity advantage** for a population *i* in a problem *F*, relative to an evaluation function φ, is given by the weighted sum of the probabilities that this population dominates the others, where the weights are the evaluations of those populations. Formally:

$$A_i\left(F, \varphi\right) = \sum_{j \neq i} p_{ij} \varphi_j(F)$$

The corresponding **opportunity cost** is given by the sum of the probabilities that other populations dominate *i*, times the evaluation of *i*. That is,

$$K_i\left(F,\varphi\right)=\varphi_i(F)\sum_{j\neq i}p_{ji}$$

We define **the worth** as the evaluation function ω that associates to each problem *F* the willingness to pay values of the underlying "densities". That is, is the vector of values that equalize, for each distribution, the opportunity advantage and the opportunity cost. We call the **worth vector** to the evaluation so obtained. The *i*th component of the worth vector is thus given by:

$$\omega_i(F)=\frac{\sum_{j\neq i}p_{ij}\omega_j(F)}{\sum_{j\neq i}p_{ji}},\quad i=1,2,\ldots,g$$

Note that, by definition, the worth has a degree of freedom regarding the choice of units.

We now show that, for each problem *F*, the worth vector exists and it is positive.

Theorem 1: *Let F be a problem. Then there exists a vector* $\mathbf{v}^*\in\boldsymbol{R}_+^g$ *such that:*

$$v_i^*=\frac{\sum_{j\neq i}p_{ij}v_j^*}{\sum_{j\neq i}p_{ji}},\quad i=1,2,\ldots,g$$

Proof

Consider the function $\varphi : V\rightarrow\boldsymbol{R}^g$, with $V=\{x\in\boldsymbol{R}_+^g \;/\; \sum_{i=1}^g x_i=g\}$, given by:

$$\varphi_i\left(\mathbf{v}\right)=v_i-\frac{1}{g-1}\left(v_i\sum_{j\neq i}p_{ji}-\sum_{j\neq i}p_{ij}v_j\right)$$

As $\sum_{j\neq i}p_{ji}\leq g-1$, we have:

$$\varphi_i\left(\mathbf{v}\right)\geq v_i-v_i+\frac{1}{g-1}\sum_{j\neq i}p_{ij}v_j\geq 0$$

Moreover,

$$\sum_{i=1}^g\varphi_i\left(\mathbf{v}\right)=g-\frac{1}{g-1}\left(\sum_{i=1}^g v_i\sum_{j\neq i}p_{ji}-\sum_{i=1}^g\sum_{j\neq i}p_{ij}v_j\right)$$

Note that, by construction, $\sum_{i=1}^g v_i\sum_{j\neq i}p_{ji}=\sum_{i=1}^g\sum_{j\neq i}p_{ij}v_j$, which means that $\sum_{i=1}^g\varphi_i\left(\mathbf{v}\right)=g$. That is, function φ maps *V* into itself. As it is a continuous function and *V* is a compact convex set, Brouwer's Theorem (e.g. Zeidler (1986)), ensures the existence of a fixpoint, $\mathbf{v}^*=\varphi(\mathbf{v}^*)$. That is,

$$v_i^* = \frac{\sum_{j \neq i} p_{ij} v_j^*}{\sum_{j \neq i} p_{ji}}, \quad i = 1, 2, \ldots, g$$

Q.e.d.

A simple sufficient condition for a solution to be unique and strictly positive is that $p_{ij} > 0$ for all i, j. Yet this is much stronger than required. What is needed to ensure those additional properties is the irreducibility of the matrix of domination probabilities. This point is discussed in Herrero & Villar (2013).

2.2 Domination probabilities and willingness to pay

We had observed in Section 1, when presenting the willingness to pay criterion in the simplest context of evaluating two urns, *A* and *B*, that the quotient $v_A^* / v_B^* = p_{AB} / p_{BA}$ measures the relative desirability of *A* with respect to *B*. In the two-population case, therefore, the willingness to pay criterion corresponds, precisely, to that of domination probabilities (except for the choice of units). The willingness to pay criterion, and hence the worth, can thus be regarded as a transitive extension of the domination probabilities evaluation. A natural question is now: How can we link the case of two populations, where the worth and the domination probabilities criteria coincide, with the general case? Or, put differently, what are the principles that the worth incorporates in order to go from two to many in a transitive way?

We shall show here that the worth is the only mapping that satisfies two intuitive and elementary principles: binariness and independence. Binariness says that when there are only two distributions being compared, then their relative evaluation should coincide with the ratio of their corresponding domination probabilities. Independence refers to the case of g > 2 distributions and is a weakening of the following idea: the ratio between the evaluation of a distribution and its opportunity advantage does not change when other populations are merged or split differently. So independence is the key property that induces the extension from two to many.

Let us be more formal regarding those notions. Let φ stand for an evaluation function and denote by $\boldsymbol{F}^g$ the set problems involving the evaluation of g "densities". The first property, **binariness**, says that, for each $F \in \boldsymbol{F}^2$,

$$\frac{\varphi_1(F)}{\varphi_2(F)} = \frac{p_{12}}{p_{21}}$$

This property conveys the idea that the domination probability is the key element for the evaluation and that, indeed, it is all we need in the simplest case in which we compare only two distributions.

For a problem $F \in \boldsymbol{F}^g$, for $g > 2$, call $F^{(i)}$ the two-population problem consisting of population *i* and the population G_{-i} that obtains from merging all other populations into a single one. Let $\varphi_i\left(F^{(i)}\right)$, $\varphi_{-i}\left(F^{(i)}\right)$ denote the evaluation of populations *i* and G_{-i} in problem $F^{(i)}$, respectively. The property of **independence** says the following:

$$\frac{\varphi_i(F)}{\varphi_i(F^{(i)})} = \frac{A_i(F,\varphi)}{A_i(F^{(i)},\varphi)}$$

That is, the ratio between the value attached to *i* in the original problem *F* and that in the modified problem $F^{(i)}$ corresponds, precisely, to the ratio of its opportunity advantages in both scenarios. The idea behind this principle is that changing the way of conforming the populations should not affect the essence of the evaluation.

The following result is obtained:

Theorem 2: *There is a unique evaluation function that satisfies binariness and independence. It is the function that attaches its worth to each group.*

Proof

Let φ be an evaluation function that satisfies those two properties. For $F \in \boldsymbol{F}^2$ binariness and a suitable normalization yield the desired result. Take now the case of $F \in \boldsymbol{F}^2$, $g > 2$, and consider the problem $F^{(i)} \in \boldsymbol{F}^2$. By binariness,

$$\frac{\varphi_i(F^{(i)})}{\varphi_{-i}(F^{(i)})} = \frac{\sum_{j\neq i} p_{ij}}{\sum_{j\neq i} p_{ji}} \qquad [2]$$

By independence, and bearing in mind the definition of opportunity advantage,

$$\frac{\varphi_i(F)}{\varphi_i(F^{(i)})} = \frac{A_i(F,\varphi)}{A_i(F^{(i)},\varphi)} \Rightarrow \varphi_i(F) = \varphi_i(F^{(i)}) \frac{\sum_{j\neq i} p_{ij}\varphi_j(F)}{\left(\sum_{j\neq i} p_{ji}\right)\varphi_{-i}(F^{(i)})}$$

Substituting according to equation [2] yields:

$$\varphi_i(F) = \varphi_{-i}(F^{(i)}) \frac{\sum_{j\neq i} p_{ij}}{\sum_{j\neq i} p_{ji}} \frac{\sum_{j\neq i} p_{ij}\varphi_j(F)}{\left(\sum_{j\neq i} p_{ji}\right)\varphi_{-i}(F^{(i)})} = \frac{\sum_{j\neq i} p_{ij}\varphi_j(F)}{\sum_{j\neq i} p_{ji}}$$

So, a mapping that satisfies binariness and independence must have this format. Theorem 1 shows that this mapping is well defined, that is, for each problem there is a vector $\mathbf{v}^*$ with this property. Therefore, we conclude that there exist a unique mapping ω

that satisfies those properties and that is such that, for all $i = 1,2,\dots,g$,

$$\omega_i(F) = \frac{\sum_{j\neq i} p_{ij}\omega_j(F)}{\sum_{j\neq i} p_{ji}}$$

Q.e.d.

3 Discussion

The worth is an endogenous, cardinal, transitive and complete criterion that enables evaluating distributions (either discrete or continuous) of monotone attributes. It provides a quantitative assessment on the relative desirability of each distribution in terms of the likelihood of getting better results. The key value judgement is that of the domination probabilities in binary comparisons (the probability that a random extraction from population A yields a better outcome than one from population B, vis a vis the opposite). The worth corresponds of a consistent application of the willingness to pay principle and turns out to agree with the (first order) stochastic dominance criterion, in the sense that if distribution A stochastically dominates distribution B, then the worth of A is larger than that of B. The worth, though, yields a complete ranking of the distributions and extends stochastic dominance through a venue that departs from that of higher order dominance.

The characterization of this criterion presented above permits one to regard the worth as a transitive extension of the evaluation based on domination probabilities, which derives from ensuring that merging the groups does not alter the relation between the evaluation and the opportunity advantage of each distribution.

3.1 Categories and numerical variables

There is an aspect of this evaluation criterion that deserves some consideration as it affects the extent of its applicability. It refers to the limited amount of information required: the matrix of domination probabilities. This is the reason why we can evaluate situations involving categorical data, as the distribution of the elements of the population into the different categories is all we need to calculate the domination probabilities. And also the motive for a warning when dealing with numerical variables, either discrete of continuous: they are to be interpreted as indexing attributes rather than as genuinely quantitative values. In particular, one has to bear in mind, when dealing with numerical variables, that this evaluation procedure does not compute the differences in the magnitude of the prizes, but just their ranking. That is, if we change the reference problem by taking the log of the prizes (or any monotone transformation) for all distributions, the worth vector will not change. Consequently, when the size of the prize differences constitutes a relevant part of the evaluation problem, a different criterion may be needed (e.g. a criterion computing the mean and variance of the distributions).

The evaluation of monetary lotteries may serve to illustrate this point. A monetary lottery can be identified with a probability distribution on a set of monetary outcomes. The standard model of choice is that in which the evaluation U of a lottery L is identified with the expectation of its outcomes. That is, assuming a discrete number of possible outcomes, $x_1, x_2, \ldots, x_g$,

$$U(L) = \sum_{i=1}^{g} \pi_i u(x_i)$$

where π_i is the probability of outcome i and $u(x_i)$ the evaluation of this particular outcome. Notice that comparing consistently alternative lotteries requires this mapping to be linear (i.e. we can change the origin and units of the elements but not applying more general transformations). Consider again the example in Table 1 and suppose that money prizes are 4, 3, 2 and 1, corresponding to categories I, II, III and IV, respectively. For any concave function u, we find that A is better than B (concavity implies that for distributions with the same mean we rank first those with a smaller variance). As the size of the prizes enters the evaluation, the ranking may differ (and in this case it actually does) from the one yielded by the worth.

This in turn suggests an immediate application of the worth: evaluating monetary lotteries when utilities are ordinal. Suppose now we have a consumer with ordinal preferences who has to choose between several monetary lotteries. We can think of money prizes as representations of utilities, so that any monotone transformation of those prizes corresponds to an alternative representation of the consumer's utility. In this context the worth principle permits the consumer to evaluate the lotteries in terms of the domination probabilities of the underlying variable (utility). The consumer ranks first the lottery for which the probability of achieving a higher utility is larger.

Note that when there is a single good (money) all increasing utility functions are ordinally equivalent. So, in this context, the worth criterion is consistent with the evaluation of lotteries of any individual endowed with ordinal monotonic preferences. The interpretation of the worth as the willingness to pay for a lottery makes here full sense.

Also observe that the case of ordinal utilities can be also interpreted in terms of lotteries with prizes of unknown magnitude (i.e. we only know that each state of the world has associated a payment which may be larger or smaller than that of another one, but do not know the size of the differences).

When we deal with numerical variables the categories are usually defined in terms of intervals over the range of the variable under consideration. The way of defining those intervals affects the evaluation outcomes because all observations that belong to the same cell are indistinguishable, as we only take into account the distribution of the population within the intervals. In some cases there is a natural way of defining those intervals, as the categories are simply parameterized by the values of the variable. This is the case, for instance, with the levels of competence that appear in the PISA studies. The OECD defines six levels of competence, based on the ability to deal with different degrees of complexity, which are parameterized in terms of values of the test scores (see Herrero, Méndez & Villar (2014) for an application to this scenario). In other cases, however, the construction of those intervals may become rather arbitrary and hence the evaluation lack of robustness. In that context the recourse to identifying each point of the support with a category or using continuous distributions may help, as it avoids the need of building those intervals. One may use standard procedures to estimate the density (e.g. kernel estimation with an optimization algorithm to determine de proper bandwidth), and then make the evaluation using the continuous version of the formula to calculate the domination probabilities, without introducing unjustified parameters.

Besides those evaluation problems in which we deal with categories *prima facie*, there are also problems in which we have to evaluate distributions of numerical variables that, nevertheless, represent levels of achievement of some attribute rather than cardinal measures. An obvious case is that in which the numbers represent ordinal utilities regarding the individual evaluation of a given commodity (e.g. income, that will be specifically considered later on). But there are many other examples. One is that of visual analogic scales or numerical rating scales that are used in medical treatments in order to assess aspects pain intensity (Hawker et al (2011), Villar (2014)). In a similar vein, but in a rather different context, we find the assignment of points to levels of a scale in some of the variables that appear in the recent EurLife study (an interactive database on quality of life in Europe, offering data drawn from the Foundation's own surveys and from other published sources). For instance, the quality of the national public education system is evaluated by given a ten level scoring rule that assigns 1 point to 'Very poor quality' and 10 points to 'Very high quality'. Note that in these cases the evaluation in terms of means or more complex functions of those variables are fully dependent on the scoring rule, which may be rather arbitrary. The worth does not need of those scores to get the evaluation.

3.2 The "social worth"

There is an interesting interpretation of the worth in terms of social choice. Consider a social choice problem involving a society with *m* agents and a set of *n* social alternatives. Each agent ranks those alternatives from top to bottom and this information is collected into a matrix **A** of relative frequencies whose generic entry, a_{ir}, tells us the share of agents who rank alternative *i* in position *r* (a square matrix). The social choice problem consists of transforming this matrix into a social ranking by some aggregation function. In this context we can think of the social alternatives as playing the role of the urns and the rank order that of the categories. Therefore, the domination probability p_{ij} describes the likelihood that an agent picked at random places alternative *i* in a higher position than alternative *j*.

The worth of social alternatives, which we call the ***social worth***, will translate the frequencies of the individual rankings into a social evaluation. We can think here of $A_i(F,\varphi)$ as the **social benefit** of choosing alternative *i* in problem *F*, relative to a social evaluation function φ, and of $K_i(F,\varphi)$ as the corresponding **social cost**. The social worth of an alternative *i* corresponds to the number w_i that equalizes the corresponding social benefits and costs and can be regarded as the social willingness to pay for getting that alternative, given the evaluations of other alternatives and their ranking by people.

Note that the social worth has elements in common with those procedures proposed by Condorcet and Borda. As in Condorcet, it is based on pairwise comparisons of the relative support of the different alternatives. As in Borda, the social worth always selects a winning option, not necessarily unique, and provides a cardinal ranking of the alternatives under consideration. Yet it differs from both of them, as illustrated by the following example (a standard example used to illustrate the differences between voting rules).

Consider a society made of 21 agents who have to rank four alternatives, A, B, C, D. Table 3 describes the ranking of the alternatives by those agents.

Table 3: Ranking of four alternatives by 21 agents

7 agents	6 agents	5 agents	3 agents
B	C	A	A
D	B	C	B
C	D	B	C
A	A	D	D

Alternative A is the one with the higher number of first positions (the winning option according to the rule known as *plurality voting*). Alternative B is the one selected by the Borda rule (and, in this particular case, by any *scoring rule*).[2] Alternative C is the Condorcet winner. Calculating the worth vector yields the following result (normalizing the mean social worth equal to one): 0.58 for alternative A, 1.91 for alternative B, 1.18 for alternative C, and 0.33 for alternative D.

From this result it clearly follows that the social worth differs from the Condorcet criterion when it comes to selecting the best social alternative. One may wonder whether the worth can be regarded as a weak form of scoring rule that assigns endogenously some scores to the different categories, even if those scores may change from one to another problem. In that case one could always express the social worth of an alternative as the weighted sum of the distribution of the population by positions, where the weights are the implicit scores of those positions. That is, $\omega_i(F) = \sum_{r=1}^{s} a_{ir}\alpha_r$, where $\alpha_r \geq 0$ is the score that the worth implicitly attaches to category *r*, and a_{ir} is the share of agents that place alternative *i* in position *r*.

Solving the implicit weights that the rule attaches to the different positions of the ranking in the example of Table 3 we find:

$$\alpha_1 = 2.957,\ \alpha_2 = 2.444,\ \alpha_3 = -0.518,\ \alpha_4 = -0.883$$

So there is no solution with positive weights for all categories in this case in which *all* scoring rules select the same winner. Moreover, it is easy to produce examples in which the implicit weights are inconsistent with the given ranking of the categories (e.g. category three gets a higher score than category two). The reason is that this is an evaluation

[2] A scoring rule is an aggregation procedure based on the support that each alternative receives (i.e. how many people rank *i* above *j*) and on the scores attached to each position in the ranking. Those scores are exogenously given, so that the resulting social ranking will depend on those parameters. The Borda rule is probably the best known of those rules.

procedure is *relative* and uses the information of the distribution of the agents support of all alternatives in order to calculate each single value.

In summary: the social worth is not a scoring rule[3] and differs from the standard choice functions as the Condorcet winner or plurality voting. Yet it is an arrovian social choice rule as it provides a complete and transitive social ranking based on individual ordinal preferences, exclusively. The social worth satisfies universal domain, the Pareto principle and anonymity, besides many other properties, but not independence of irrelevant alternatives. The characteristic feature of this rule is that the evaluation of each alternative depends on how many people rank this alternative higher than the others, how those alternatives are in turn ranked with respect to the others, and so on and so forth.

It is interesting to observe that in this context the domination principle turns out to be a property reminiscent of the ideas of Morales (a Spanish thinker who lived during the times of Borda and Condorcet), who claimed that ranking social alternatives should be related to the "amount of opinion" of the citizens. Indeed, we can interpret the property of binariness as an expression of this notion, because when there are only two alternatives their valuations should be proportional to the number of agents who favour each of them.

[3] Young (1975) proves that all scoring rules satisfy the property of reinforcement. This property says that if an option is chosen as the best one in two different societies, then it will also be chosen by the union of both societies. The social worth does not satisfy this property as the evaluation of each alternative depends on how it relates with all others, and this relationship may change when combining two different societies.

REFERENCES

1. Bellù, L.G. & Liberati, P. (2005), Social Welfare Analysis of Income Distributions Ranking Income Distributions with Crossing Generalised Lorenz Curves, Food and Agriculture Organization of the United Nations.
2. Bourguignon, F., Ferreira, F.H.G. & Leite, P.G. (2007), Beyond Oaxaca–Blinder: Accounting for differences in household income distributions, **Journal of Economic Inequality**, DOI 10.1007/s10888-007-9063-y.
3. Crespo, J.A., Li, Y. & Ruiz–Castillo, J. (2013), The Measurement of the Effect on Citation Inequality of Differences in Citation Practices across Scientific Fields. **PLoS ONE** 8(3): e58727. doi:10.1371/journal.pone.0058727.
4. Cuhadaroglu, T. (2013), My group beats your group : evaluating non-income inequalities, w.p. School of Economics and Finances, U. Of St. Andrews.
5. Gonzalez-Diaz, J., Hendrichx, R., & Lohmann, E (2013), Paired comparison analysis: an axiomatic approach to ranking methods, **Social Choice and Welfare,** in press.
6. Grannis, R. (2002), Segregation Indices and their Functional Inputs, **Sociological Methodology**, 32 (1), 69-84.
7. Hawker, G.A., Mian, S., Kendzerska, T. & French, M. (2011), Measures of Adult Pain, **Arthritis Care & Research**, Vol. 63, No. S11, pp S240–S252 DOI 10.1002/acr.20543.
8. Herrero, C., Méndez, I. & Villar, A. (2014), Analysis of group performance with categorical data when agents are heterogeneous: The evaluation of scholastic performance in the OECD through PISA, *Economics of Education Review*, vol. 40 : 140-151.
9. Herrero, C. & Villar, A. (2013), On the Comparison of Group Performance with Categorical Data. **PLoS ONE** 8(12): e84784. doi:10.1371/journal.pone.0084784.
10. Laband, D.N., & Piette, M.J. (1994). The relative impacts of economics journals: 1970-1990, **Journal of Economic Literature**, 32(2), 640-666.
11. Laslier, J. (1997), **Tournament solutions and majority voting,** Springer, Berlin, Heildelberg, New York.
12. Li, F., Yi, K, & Jestes, J. (2009), Ranking Distributed Probabilistic Data, ***SIGMOD'09,*** June 29–July 2.
13. Lieberson, S. (1976), Rank-sum comparisons between groups. **Sociological Methodology** 7: 276–291. doi: 10.2307/270713
14. Martínez-Mekler G, Martínez RA, del Río MB, Mansilla R, Miramontes P, et al. (2009),

Universality of Rank-Ordering Distributions in the Arts and Sciences. **PLoS ONE** 4(3): e4791. doi:10.1371/journal.pone.0004791.
15. Palacios-Huerta, I. & Volij, O (2004), The Measurement of Intellectual Influence, **Econometrica,** 72(3): 963-977.
16. Pinski, G., & Narin, F. (1976). Citation influence for journal aggregates of scientific publications: Theory, with application to the literature of physics, **Information Processing and Management**, 12(5), 297--312.
17. Reardon, S. F. & Firebaugh, G. (2002), Measures of Multi-Group Segregation, **Sociological Methodology**, 32 : 33-76.
18. Rosvall, M. & Bergstrom, C.T. (2007), An information-theoretic framework for resolving community structure in complex networks, **Proceedings of the National Academy of Sciences**, doi/10.1073/pnas.0611034104.
19. Sheriff, G. & Maguire, K. (2013) Ranking Distributions of Environmental Outcomes Across Population Groups,
20. Shorrocks, A. (1983),
21. Slutzki, G., & Volij, O. (2006), Scoring of web pages and tournaments, **Social Choice and Welfare,** 26 (1): 75-92.
22. Villar, A. (2014),
23. Waltman, L. & van Eck, N.J. (2010), The Relation Between Eigenfactor, Audience Factor, and Influence Weight, **Journal of the American Society for Information, Science and Technology**, 61 : 1476-1486.
24. Yalonetzky, G. (2012), A Dissimilarity Index of Multidimensional Inequality of Opportunity, **The Journal of Economic Inequality**, 10(3): 343-373.
25. Young, P. (1975)
26. Zeidler, E. (1986), **Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems**, Springer- Verlag New York.